Basic Mathematics

Dimensional Analysis

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The aim of this package is to provide a short self assessment programme for students who wish to learn how to use dimensional analysis to investigate scientific equations.

Last Revision Date: June 2, 2004 Version 1.0

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction

It is important to realise that it only makes sense to add the same sort of quantities, e.g. **area may be added to area** but **area may not be added to temperature!** These considerations lead to a powerful method to analyse scientific equations called **dimensional analysis.**

One should note that while **units are arbitrarily chosen** (an alien civilisation will not use seconds or weeks), **dimensions represent fundamental quantities** such as time.

Basic dimensions are written as follows:

Dimension	Symbol
Length	L
Time	T
Mass	M
Temperature	K
Electrical current	I

See the package on **Units** for a review of SI units.

Example 1 An area can be expressed as a length times a length. Therefore the **dimensions of area** are $\mathrm{L} \times \mathrm{L} = \mathrm{L}^2$. (A given area could be expressed in the SI units of square metres, or indeed in any appropriate units.) We sometimes write: **[area]**$= \mathrm{L}^2$
In some equations symbols appear which do **not** have any associated dimension, **e.g.,** in the formula for the area of a circle, πr^2, π is just a number and does **not** have a dimension.

EXERCISE 1. Calculate the dimensions of the following quantities (click on the **green** letters for the solutions).

(a) Volume
(b) Speed
(c) Acceleration
(d) Density

Quiz Pick out the units that have a different dimension to the other three.

(a) $\mathrm{kg\,m^2\,s^{-2}}$
(b) $\mathrm{g\,mm^2\,s^{-2}}$
(c) $\mathrm{kg^2\,m\,s^{-2}}$
(d) $\mathrm{mg\,cm^2\,s^{-2}}$

2. Checking Equations

Example 2 Consider the equation

$$y = x + \frac{1}{2}kx^3$$

Since any terms which are added together or subtracted **MUST** have the same dimensions, in this case y, x and $\frac{1}{2}kx^3$ have to have the same dimensions.

We say that such a scientific equation is **dimensionally correct**. (If it is not true, the equation must be wrong.)

If in the above equation x and y were both lengths (dimension L) and 1/2 is a dimensionless number, then for the $\frac{1}{2}kx^3$ term to have the same dimension as the other two, we would need:

$$\text{dimension of } k \times \mathrm{L}^3 = \mathrm{L}$$

$$\therefore \text{dimension of } k = \frac{\mathrm{L}}{\mathrm{L}^3} = \mathrm{L}^{-2}$$

So k would have dimensions of one over area, i.e., $[k] = \mathrm{L}^{-2}$.

Quiz Hooke's law states that the force, F, in a spring extended by a length x is given by $F = -kx$. From Newton's second law $F = ma$, where m is the mass and a is the acceleration, calculate the dimension of the spring constant k.

(a) $\mathrm{M\,T^{-2}}$ (b) $\mathrm{M\,T^{2}}$ (c) $\mathrm{M\,L^{-2}\,T^{-2}}$ (d) $\mathrm{M\,L^{2}\,T^{2}}$

Example 3 The expressions for **kinetic energy** $E = \frac{1}{2}mv^2$ (where m is the mass of the body and v is its speed) and **potential energy** $E = mgh$ (where g is the acceleration due to gravity and h is the height of the body) look very different but both describe energy. One way to see this is to note that they have the same dimension.

Dimension of **kinetic energy**

$$\begin{aligned} \tfrac{1}{2}mv^2 &\Rightarrow \mathrm{M\,(L\,T^{-1})^2} \\ &= \mathrm{M\,L^2\,T^{-2}} \end{aligned}$$

Dimension of **potential energy**

$$\begin{aligned} mgh &\Rightarrow \mathrm{M\,(L\,T^{-2})\,L} \\ &= \mathrm{M\,L^2\,T^{-2}} \end{aligned}$$

Both expressions have the same dimensions, they can therefore be added and subtracted from each other.

EXERCISE 2. Check that the dimensions of each side of the equations below agree (click on the **green** letters for the solutions).

(a) The volume of a cylinder of radius r and length h $V = \pi r^2 h$.

(b) $v = u + at$ for an object with initial speed u, (constant) acceleration a and final speed v after a time t.

(c) $E = mc^2$ where E is energy, m is mass and c is the speed of light.

(d) $c = \lambda\nu$, where c is the speed of light, λ is the wavelength and ν is the frequency

Note that dimensional analysis is a way of checking that equations might be true. It does not prove that they are definitely correct. Dimensional analysis would suggest that both Einstein's equation $E = mc^2$ and the (incorrect) equation $E = \frac{1}{2}mc^2$ might be true. On the other hand dimensional analysis shows that $E = mc^3$ makes no sense.

3. Dimensionless Quantities

Some quantities are said to be dimensionless. These are then pure numbers which would be the same no matter what units are used (e.g., the mass of a proton is roughly 1850 times the mass of an electron no matter how you measure mass).

Example 4 The ratio of one mass m_1 to another mass m_2 is dimensionless:

$$\text{dimension of the fraction } \frac{m_1}{m_2} = \frac{\mathrm{M}}{\mathrm{M}} = 1$$

The dimensions have canceled and the result is a number (which is independent of the units, i.e., it would be the same whether the masses were measured in kilograms or tonnes).

Note that angles are defined in terms of ratios of lengths. They are therefore dimensionless! Functions of dimensionless variables are themselves dimensionless.

Very many functions are dimensionless. The following quantities are **important cases of dimensionless quantities:**

Trigonometric functions	Logarithms
Exponentials	Numbers, e.g., π

Note the following properties:

- Functions of dimensionless variables are dimensionless.
- Dimensionless functions must have dimensionless arguments.

Quiz If the number of radioactive atoms is found to be given as a function of time t by

$$N(t) = N_0 \exp(-kt)$$

where N_0 is the number of atoms at time $t = 0$, what is the dimension of k?

(a) LT (b) log(T) (c) T (d) T^{-1}

EXERCISE 3. Determine the dimensions of the expressions below (click on the **green** letters for the solutions).

(a) In a Young's slits experiment the angle θ of constructive interference is related to the wavelength λ of the light, the spacing of the slits d and the order number n by $d\sin(\theta) = n\lambda$. Show that this is dimensionally correct.

(b) The Boltzmann distribution in thermodynamics involves the factor $\exp(-E/(kT))$ where E represents energy, T is the temperature and k is Boltzmann's constant. Find the dimensions of k.

Quiz Use dimensional analysis to see which of the following expressions is allowed if P is a pressure, t is a time, m is a mass, r is a distance, v is a velocity and T is a temperature.

(a) $\log\left(\frac{Pt}{mr}\right)$ (b) $\log\left(\frac{Prt^2}{m}\right)$ (c) $\log\left(\frac{Pr^2}{mt^2}\right)$ (d) $\log\left(\frac{Pr}{mtT}\right)$

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. Newton's law of gravity states that the gravitational force between two masses, m_1 and m_2, separated by a distance r is given by $F = Gm_1m_2/r^2$. What are the dimensions of G?
(a) $\mathrm{L}^3\,\mathrm{M}^{-1}\,\mathrm{T}^{-2}$ (b) $\mathrm{M}^2\,\mathrm{L}^{-2}$ (c) $\mathrm{M\,L\,T}^{-2}$ (d) $\mathrm{M}^{-1}\,\mathrm{L}^{-3}\,\mathrm{T}^2$

2. The coefficient of thermal expansion, α of a metal bar of length ℓ whose length expands by $\Delta\ell$ when its temperature increases by $\Delta\mathrm{T}$ is given by $\Delta\ell = \alpha\ell\Delta\mathrm{T}$. What are the dimensions of α?
(a) K^{-1} (b) $\mathrm{L}^2\mathrm{T}^{-1}$ (c) $\mathrm{L}^2\mathrm{T}^{-1}$ (d) $\mathrm{L}^{-2}\mathrm{K}^{-1}$

3. The position of a mass at the end of a spring is found as a function of time to be $A\sin(\omega t)$. Select the dimensions of A and ω.
(a) L & T (b) L & Dimensionless
(c) sin(L) & T^{-1} (d) L & T^{-1}

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) A volume is given by multiplying three lengths together:

$$\begin{aligned} \text{Dimension of volume} &= \mathrm{L} \times \mathrm{L} \times \mathrm{L} \\ &= \mathrm{L}^3 \end{aligned}$$

So $[\textbf{volume}] = \mathrm{L}^3$
(The SI units of volume are cubic metres.)

Click on the **green** square to return

□

Exercise 1(b) Speed is the rate of change of distance with respect to time.

$$\text{Dimensions of speed} = \frac{\mathrm{L}}{\mathrm{T}} = \mathrm{L\,T^{-1}}$$

So $[\textbf{speed}] = \mathrm{L\,T^{-1}}$
(The SI units of speed are metres per second.)

Click on the **green** square to return

□

Exercise 1(c) Acceleration is the rate of change of speed with respect to time

$$\text{Dimensions of acceleration} = \frac{\mathrm{L\,T^{-1}}}{\mathrm{T}} = \mathrm{L\,T^{-2}}$$

So $[\textbf{acceleration}] = \mathrm{L\,T^{-2}}$
(The SI units of acceleration are metres per second squared.)

Click on the **green** square to return □

Exercise 1(d) Density is the mass per unit volume, so using the dimension of volume we get:

$$\text{Dimensions of volume} = \frac{\mathrm{M}}{\mathrm{L}^3} = \mathrm{M\,L^{-3}}$$

So $[\textbf{density}] = \mathrm{M\,L^{-3}}$
(The SI units of density are $\mathrm{kg\,m^{-3}}$.)

Click on the **green** square to return □

Exercise 2(a) We want to check the dimensions of $V = \pi r^2 h$. We know that the dimensions of volume are $[\text{volume}] = \mathrm{L}^3$. The right hand side of the equation has dimensions:

$$\text{dimensions of } \pi r^2 h = \mathrm{L}^2 \times \mathrm{L} = \mathrm{L}^3$$

So both sides have the dimensions of volume.

Click on the **green** square to return □

Exercise 2(b) We want to check the dimensions of the equation $v = u + at$. Since v and u are both speeds, they have dimensions $\mathrm{L\,T^{-1}}$. Therefore we only need to verify that at has this dimension. To see this consider:

$$[at] = (\mathrm{L\,T^{-2}}) \times \mathrm{T} = \mathrm{L\,T^{-2}} \times \mathrm{T} = \mathrm{L\,T^{-1}}$$

So the equation is dimensionally correct, and all the terms have dimensions of speed.

Click on the **green** square to return

□

Exercise 2(c) We want to check the dimensions of the equation $E = mc^2$. Since E is an energy it has dimensions $\mathrm{M\,L^2\,T^{-2}}$. The right hand side of the equation can also be seen to have this dimension, if we recall that m is a mass and c is the speed of light (with dimension $\mathrm{L\,T^{-1}}$). Therefore

$$\begin{aligned} [mc^2] &= \mathrm{M\,(L\,T^{-1})^2} \\ &= \mathrm{M\,L^2\,T^{-2}} \end{aligned}$$

So the equation is dimensionally correct, and all terms that we add have dimensions of energy.

Click on the **green** square to return □

Exercise 2(d) We want to check the dimensions of the equation $c = \lambda\nu$. Since c is the speed of light it has dimensions $\mathrm{L\,T^{-1}}$. The right hand side of the equation involves wavelength $[\lambda] = \mathrm{L}$ and frequency $[\nu] = \mathrm{T}^{-1}$. We thus have

$$[\lambda\nu] = \mathrm{L} \times \mathrm{T}^{-1}$$

which indeed also has dimensions of speed, so the equation is dimensionally correct.

Click on the **green** square to return

□

Exercise 3(a) We want to check the dimensions of $d\sin(\theta) = n\lambda$. Both d and λ have dimensions of length. The angle θ and $\sin(\theta)$ as well as the number n must all be dimensionless. Therefore we have

$$\begin{aligned} \mathrm{L} \times 1 &= 1 \times \mathrm{L} \\ \therefore [d\sin(\theta)] &= [n\lambda] \end{aligned}$$

So both sides have dimensions of length.

Click on the **green** square to return □

Exercise 3(b) The factor $\exp(-E/(kT))$ is an exponential and so must be dimensionless. Therefore its argument $-E/(kT)$ must also be dimensionless.

The minus sign simply corresponds to multiplying by minus one and is dimensionless. Energy, E has dimensions $[E] = \mathrm{M\,L^2\,T^{-2}}$ and temperature has dimensions K, so the dimensions of Boltzmann's constant are

$$[k] = \frac{\mathrm{M\,L^2\,T^{-2}}}{\mathrm{K}} = \mathrm{M\,L^2\,T^{-2}\,K^{-1}}$$

(So the SI units of Boltzmann's constant are $\mathrm{kg\,m^2\,s^{-2}\,{}^{o}C^{-1}}$).

Click on the **green** square to return

□

Solutions to Quizzes

Solution to Quiz: $\mathrm{kg}^2\,\mathrm{m}\,\mathrm{s}^{-2}$ has

$$\text{dimensions} = \mathrm{M}^2\,\mathrm{L}\,\mathrm{T}^{-2}$$

It can be checked that all the other answers have dimension $\mathrm{M}\,\mathrm{L}^2\,\mathrm{T}^{-2}$.

End Quiz

Solution to Quiz: From Hooke's law, $F = -kx$, we see that we can write

$$k = \frac{F}{x}$$

Now $F = ma$, so the dimensions of force are given by

$$\begin{aligned} [\mathrm{F}] &= \mathrm{M} \times (\mathrm{L\,T^{-2}}) \\ &= \mathrm{M\,L\,T^{-2}} \end{aligned}$$

Therefore the spring constant has dimensions

$$\begin{aligned} [k] &= \frac{\mathrm{M\,L\,T^{-2}}}{\mathrm{L}} \\ &= \mathrm{M\,T^{-2}} \end{aligned}$$

End Quiz

Solution to Quiz: In $N(t) = N_0 \exp(-kt)$, the exponential and its argument must be dimensionless. Therefore kt has to be dimensionless. Thus

$$\text{dimensions of } k \times \mathrm{T} = 1$$

So the dimension of k must be inverse time, i.e., $[k] = \mathrm{T}^{-1}$.

End Quiz

Solution to Quiz: First note that the argument of a logarithm must be dimensionless. Now pressure is force over area, so it has dimensions

$$
\begin{aligned}
[P] &= \frac{\mathrm{M\,L\,T^{-2}}}{\mathrm{L^2}} \\
&= \mathrm{M\,L^{-1}\,T^{-2}}
\end{aligned}
$$

Therefore the combination $\frac{Prt^2}{m}$ is dimensionless since

$$
\begin{aligned}
\left[\frac{Prt^2}{m}\right] &= \frac{(\mathrm{M\,L^{-1}\,T^{-2}}) \times \mathrm{L} \times \mathrm{T^2}}{\mathrm{M}} \\
&= \frac{\mathrm{M}}{\mathrm{M}} \\
&= 1
\end{aligned}
$$

None of the other combinations are dimensionless and so it would be completely meaningless to take their logarithms. End Quiz