PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

Basic Mathematics

The UNIVERSITY of PLYMOUTH

Functions

R Horan & M Lavelle

The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic understanding of the concept of functions.

 rhoran@plymouth.ac.uk , mlavelle@plymouth.ac.uk

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1. Functions (Introduction)

The concept of a function is essential in mathematics. There are two common notations in use:

$$\begin{array}{ll} \text{(a)} & f(x) = x^2 + 2\,, \\ \text{(b)} & f : x \mapsto x^2 + 2\,. \end{array}$$

Part (a) is commonly used. Part (b) is interpreted as

the function f maps x to $x^2 + 2$.

Example 1 If two functions are given as $f(x) = 2x + 3$, and $g(x) = 3 - x^2$, then

$$\begin{array}{ll} \text{(a)} & f(2) = 2 \times 2 + 3 = 7 \\ \text{(b)} & f(-3) = 2 \times (-3) + 3 = -6 + 3 = -3 \\ \text{(c)} & g(0) = 3 - (0)^2 = 3 \\ \text{(d)} & g(4) = 3 - (4)^2 = 3 - 16 = -13 \end{array}$$

Example 2 Find the numbers which map to zero under the function

$$h : x \mapsto x^2 - 9\,.$$

Solution The function can also be written as $h(x) = x^2 - 9$ and if x maps to zero then $h(x) = 0$, i.e.

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \end{aligned}$$

since squaring both 3 and -3 gives the value 9.

Exercise 1. Two functions are given as $h : x \mapsto x^2 - 4$ and $g : x \mapsto 10x + 5$. Find the following: (Click on the green letters for solution.)

(a) $h(1)$
(b) $h(-2)$
(c) $h(0)$
(d) $g(3)$
(e) $g(-1)$
(f) the values of x such that $h(x) = 12$

2. Flow Diagrams

The function $f(x) = 2x + 3$ in **example 1** may be represented as a flow diagram.

$$\xrightarrow{x} \boxed{\text{multiply by 2}} \xrightarrow{2x} \boxed{\text{add 3}} \xrightarrow{2x+3}$$

From the diagram it is clear that the order of the operations cannot be confused. First multiply by 2 and then add 3.

Example 3 Draw a flow diagram for the two functions given by

(a) $k : x \mapsto (3x-2)^2$, (b) $g(x) = \dfrac{4x-2}{3}$.

Solution

(a) $$\xrightarrow{x} \boxed{\text{multiply by 3}} \xrightarrow{3x} \boxed{\text{subtract 2}} \xrightarrow{3x-2} \boxed{\text{square}} \xrightarrow{(3x-2)^2}$$

(b) $$\xrightarrow{x} \boxed{\text{multiply by 4}} \xrightarrow{4x} \boxed{\text{subtract 2}} \xrightarrow{4x-2} \boxed{\text{divide by 3}} \xrightarrow{\frac{4x-2}{3}}$$

Exercise 2. Draw flow diagrams for each of the following functions. (Click on the green letters for solution.)

(a) $h : x \mapsto 6x + 1$ (b) $h : x \mapsto 4(3 - 2x)$ (c) $h : x \mapsto (2x - 5)^2$

(d) $g : x \mapsto 3x^2 - 4$ (e) $g : x \mapsto \dfrac{2x^2}{3} + 5$ (f) $g : x \mapsto \sqrt{x^2 + 2}$

Now try this short quiz.

Quiz From the functions listed below choose the one which is given by the following flow diagram.

$\xrightarrow{x}$ cube $\longrightarrow$ divide by 4 $\longrightarrow$ subtract 1 $\longrightarrow$ square $\longrightarrow$ add 4 $\longrightarrow$

(a) $\left(\dfrac{x^3}{4} - 1\right)^2 + 4$ (b) $\left(\dfrac{x^3}{4} - (1)^2\right) + 4$

(c) $\left(\dfrac{x^3}{4 - 1}\right)^2 + 4$ (d) $\left(\left(\dfrac{x}{4}\right)^3 + (1)^2\right) + 4$

3. Composite Functions

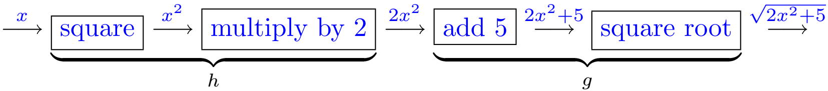
The function $f(x) = 2x + 3$, from **example 1**, is composed of two simpler functions, i.e. *multiply by 2* and *add 3*. If these two functions are written as $h : x \mapsto 2x$ and $g : x \mapsto x + 3$ then the composition of these two functions is written gh (sometimes as $g \circ h$ or $g(h(x))$).

Example 4 If $h : x \mapsto 2x^2$ and $g : x \mapsto \sqrt{x+5}$, find the composite function gh.

Solution Applying first h and then g results in the composite function

$$gh = \sqrt{2x^2 + 5}\,.$$

This can best be seen by using flow diagrams.



N.B. The composition of two functions, fg,
is *NOT* the same as the product of two functions.

Exercise 3. For each of the following functions, f, g, write down the function fg. (Click on the green letters for solution.)

(a) $f : (x) = 2x + 1,\ g(x) = x - 3$ (b) $f : x \mapsto 2x - 1,\ g : x \mapsto x^2$

(c) $f(x) = x^2,\ g(x) = 2x - 1$ (d) $f : x \mapsto x + 3,\ g : x \mapsto x - 3$

(e) $f(x) = \dfrac{x}{3} - 2,\ g(x) = 3x^2$ (f) $f : x \mapsto 3x^2,\ g : x \mapsto \dfrac{x}{3} - 2$

The functions in part (b) above are reversed in part (c). The results show that, in general, reversing the order of two functions changes the composite function. In simpler terms if h and k are two functions then, in general, $hk \neq kh$.

Quiz Two functions are given as

$$f : x \mapsto 2x^2 - 1 \text{ and } g : x \mapsto x - 3\,.$$

Which of the following is a solution to $f\,(g(x)) = 7$?

(a) $x = 5$ (b) $x = -5$ (c) $x = 2$ (d) $x = -2$

4. Inverse Functions

If a function f maps m to n then the *inverse function*, written as f^{-1}, maps n to m.

Example 5 Find the inverse of the function $h : x \mapsto \dfrac{4x-3}{2}$.

Solution First draw a flow diagram for the function.

$$\xrightarrow{x} \boxed{\text{multiply by 4}} \xrightarrow{4x} \boxed{\text{subtract 3}} \xrightarrow{4x-3} \boxed{\text{divide by 2}} \xrightarrow{\frac{4x-3}{2}}$$

Now draw a flow diagram, **starting from the right**, with each operation replaced by its inverse.

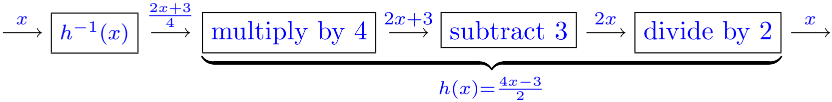
$$\xleftarrow{\frac{2x+3}{4}} \boxed{\text{divide by 4}} \xleftarrow{2x+3} \boxed{\text{add 3}} \xleftarrow{2x} \boxed{\text{multiply by 2}} \xleftarrow{x}$$

The inverse of $h : x \mapsto \dfrac{4x-3}{2}$ is thus $h^{-1} : x \mapsto \dfrac{2x+3}{4}$.

If a function f has an inverse f^{-1} then the composite function ff^{-1} is the identity function which was mentioned in exercise 3(d), i.e. $ff^{-1}: x \mapsto x$. It is also true that $f^{-1}f : x \mapsto x$.

Example 6 For the function h in **example 5**, show that the composite function hh^{-1} is the identity function.

Solution First note that $h^{-1} : x \mapsto \dfrac{2x+3}{4}$. For the composition $hh^{-1}(x)$, therefore, this must be operated on by the function h, i.e. in the first flow diagram of **example 5** the input on the right hand side must be $\dfrac{2x+3}{4}$.



A similar exercise will show that $h^{-1}h$ is also the identity function.

Exercise 4. Find the inverse of each of the following functions in the form $x \mapsto \ldots$ (Click on the green letters for solution.)

(a) $f(x) = 3x + 4$

(b) $f : x \mapsto 4(x - 1)$

(c) $f(x) = 3(2x + 5)$

(d) $g : x \mapsto -8x + 3$

(e) $g(x) = \frac{1}{2}(3x + 4) + 6$

(f) $g : x \mapsto \frac{2x + 1}{3}$

Quiz Two functions are given as

$$f : x \mapsto \frac{1}{2}x \text{ and } g : x \mapsto 3x + 16\,.$$

If $h = fg$, which of the following is h^{-1} ?

(a) $h^{-1} : x \mapsto \frac{1}{3}(2x + 16)$

(b) $h^{-1} : x \mapsto 3(2x + 16)$

(c) $h^{-1} : x \mapsto \frac{1}{3}(2x - 16)$

(d) $h^{-1} : x \mapsto 2\left(\frac{x}{3} - 16\right)$

5. Quiz on Functions

Begin Quiz If three functions are given as $f : x \mapsto x^2$, $g : x \mapsto 4x$, $h : x \mapsto x + 5$, choose the correct options for the following.

1. $fg : x \mapsto \ldots$

(a) $8x^2$, (b) $4x^2$, (c) $16x$, (d) $16x^2$.

2. $gf : x \mapsto \ldots$

(a) $8x^2$, (b) $4x^2$, (c) $16x$, (d) $16x^2$.

3. $gh : x \mapsto \ldots$

(a) $5 + 4x$, (b) $4 + 5x$, (c) $4x + 5$, (d) $4(x + 5)$.

4. $(hg)^{-1} : x \mapsto \ldots$

(a) $\frac{1}{4}(x - 5)$, (b) $\frac{1}{4}(x + 5)$, (c) $\frac{1}{5}(x + 4)$, (d) $\frac{1}{5}(x - 4)$.

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) The function is $h(x) = x^2 - 4$ so

$$\begin{aligned} h(1) &= (1)^2 - 4 \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

Click on the green square to return □

Exercise 1(b) The function is $h(x) = x^2 - 4$ so

$$\begin{aligned} h(-2) &= (-2)^2 - 4 \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

Click on the green square to return □

Exercise 1(c) The function is $h(x) = x^2 - 4$ so

$$\begin{aligned} h(0) &= (0)^2 - 4 \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

Click on the green square to return □

Exercise 1(d) The function is $g(x) = 10x + 5$ so

$$\begin{aligned} g(3) &= 10 \times (3) + 5 \\ &= 30 + 5 \\ &= 35 \end{aligned}$$

Click on the green square to return □

Exercise 1(e) The function is $g(x) = 10x + 5$ so

$$\begin{aligned} g(-1) &= 10 \times (-1) + 5 \\ &= -10 + 5 \\ &= -5 \end{aligned}$$

Click on the green square to return □

Exercise 1(f) If $h(x) = 12$ then since $h(x) = x^2 - 4$,

$$\begin{aligned} x^2 - 4 &= 12 \\ x^2 &= 4 + 12 \\ &= 16 \\ x &= \pm 4 \end{aligned}$$

since the square of both 4 and -4 is 16.

Click on the green square to return □

Exercise 2(a)

For $h : x \mapsto 6x + 1$ the flow diagram is

$$\xrightarrow{x} \boxed{\text{multiply by 6}} \xrightarrow{6x} \boxed{\text{add 1}} \xrightarrow{6x+1}$$

Click on the green square to return □

Exercise 2(b)

For $h : x \mapsto 4(3-2x)$ the flow diagram is

$$\xrightarrow{x} \boxed{\text{multiply by } -2} \xrightarrow{-2x} \boxed{\text{add } 3} \xrightarrow{3-2x} \boxed{\text{multiply by } 4} \xrightarrow{4(3-2x)}$$

Click on the green square to return □

Exercise 2(c)

For $h : x \mapsto (2x-5)^2$ the flow diagram is

$$\xrightarrow{x} \boxed{\text{multiply by 2}} \xrightarrow{2x} \boxed{\text{subtract 5}} \xrightarrow{2x-5} \boxed{\text{square}} \xrightarrow{(2x-5)^2}$$

Click on the green square to return □

Exercise 2(d)
For $g : x \mapsto 3x^2 - 4$ the flow diagram is

$$\xrightarrow{x} \boxed{\text{square}} \xrightarrow{x^2} \boxed{\text{multiply by 3}} \xrightarrow{3x^2} \boxed{\text{subtract 4}} \xrightarrow{3x^2-4}$$

Click on the green square to return ☐

Exercise 2(e)

For $g : x \mapsto \dfrac{2x^2}{3} + 5$ the flow diagram is

$$\xrightarrow{x} \boxed{\text{square}} \xrightarrow{x^2} \boxed{\text{multiply by 2}} \xrightarrow{2x^2} \boxed{\text{divide by 3}} \xrightarrow{\frac{2x^2}{3}} \boxed{\text{add 5}} \xrightarrow{\frac{2x^2}{3}+5}$$

Click on the green square to return □

Exercise 2(f)

For $g : x \mapsto \sqrt{x^2+2}$ the flow diagram is

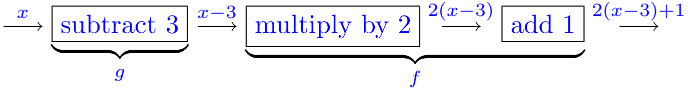
$$\xrightarrow{x} \boxed{\text{square}} \xrightarrow{x^2} \boxed{\text{add } 2} \xrightarrow{x^2+2} \boxed{\text{take square root}} \xrightarrow{\sqrt{x^2+2}}$$

Click on the green square to return □

Exercise 3(a)
For the functions $f:(x)=2x+1$, $g(x)=x-3$ the function fg is

$$fg : x \mapsto 2(x-3)+1 = 2x-6+1 = 2x-5\,.$$

The flow diagram is



The function fg can also be determined as follows. The two functions can be written as $f(z)=2z+1$ and $g(x)=x-3$. Then by substituting $z=g(x)$ into $f(z)=2z+1$,

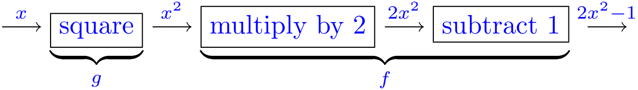
$$\begin{aligned} fg(x) &= f(g(x)) \\ &= 2g(x)+1 \\ &= 2(x-3)+1 \\ &= 2x-5\,. \end{aligned}$$

Click on the green square to return

Exercise 3(b)
For the functions $f : x \mapsto 2x - 1$, $g : x \mapsto x^2$ the function fg is

$$fg : x \mapsto 2x^2 - 1\,.$$

The flow diagram is



The composition may also be determined by writing $f(z) = 2z - 1$ and $g(x) = x^2$ and substituting $z = g(x)$, obtaining

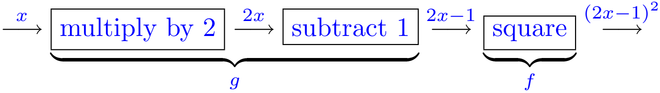
$$\begin{aligned} fg(x) &= f(g(x)) \\ &= 2g(x) + 1 \\ &= 2\left(x^2\right) - 1 \\ &= 2x^2 - 1\,. \end{aligned}$$

Click on the green square to return □

Exercise 3(c)
For the functions $f : x \mapsto x^2$, $g : x \mapsto 2x - 1$ the function fg is

$$fg : x \mapsto (2x-1)^2 \, .$$

The flow diagram is



Alternatively, writing $f(z) = z^2$ and $g(x) = 2x - 1$, then substituting for $z = g(x)$:

$$\begin{aligned} fg(x) &= f(g(x)) \\ &= (g(x))^2 \\ &= (2x-1)^2 \, . \end{aligned}$$

Click on the green square to return ☐

Exercise 3(d)

For the functions $f : x \mapsto x + 3$, $g : x \mapsto x - 3$ the function fg is

$$fg : x \mapsto x\,.$$

The flow diagram is

$$\xrightarrow{x} \underbrace{\boxed{\text{subtract } 3}}_{g} \xrightarrow{x-3} \underbrace{\boxed{\text{add } 3}}_{f} \xrightarrow{(x-3)+3}$$

The function which maps x to x is called the *identity function*. The identity function does not change the value of x.

Click on the green square to return

Exercise 3(e)

For the functions $f : x \mapsto \frac{x}{3} - 2$, $g : x \mapsto 3x^2$ the function fg is

$$fg : x \mapsto x^2 - 2\,.$$

The flow diagram is

$$\xrightarrow{x} \underbrace{\boxed{\text{square}} \xrightarrow{x^2} \boxed{\text{multiply by 3}}}_{g} \xrightarrow{3x^2} \underbrace{\boxed{\text{divide by 3}} \xrightarrow{x^2} \boxed{\text{subtract 2}}}_{f} \xrightarrow{x^2-2}$$

Writing $f(x) = \frac{z}{3} - 2$ and $g(x) = 3x^2$ and substituting $z = g(x)$,

$$\begin{aligned} fg(x) = f(g(x)) &= \frac{g(x)}{3} - 2 \\ &= \frac{3x^2}{3} - 2 \\ &= x^2 - 2\,. \end{aligned}$$

Click on the green square to return □

Exercise 3(f)

For the functions $f : x \mapsto 3x^2$, $g : x \mapsto \dfrac{x}{3} - 2$ the function fg is

$$fg : x \mapsto 3\left(\frac{x}{3} - 2\right)^2 .$$

The flow diagram is

$$\xrightarrow{x} \underbrace{\boxed{\text{divide by 3}} \xrightarrow{\frac{x}{3}} \boxed{\text{subtract 2}}}_{g} \xrightarrow{\frac{x}{3}-2} \underbrace{\boxed{\text{square}} \xrightarrow{\left(\frac{x}{3}-2\right)^2} \boxed{\text{multiply by 3}}}_{f} \xrightarrow{3\left(\frac{x}{3}-2\right)^2}$$

Alternatively, the functions are $f(z) = 3z^2$ and $g(x) = \dfrac{x}{3} - 2$. Thus

$$\begin{aligned} fg(x) &= f(g(x)) \\ &= 3(g(x))^2 \\ &= 3(\frac{x}{3} - 2)^2 . \end{aligned}$$

Click on the green square to return □

Exercise 4(a)
For the function $f : x \mapsto 3x + 4$ the flow diagram is

$$\xrightarrow{x} \boxed{\text{multiply by 3}} \xrightarrow{3x} \boxed{\text{add 4}} \xrightarrow{3x+4}$$

The inverse is thus

$$\xleftarrow{\frac{x-4}{3}} \boxed{\text{divide by 3}} \xleftarrow{x-4} \boxed{\text{subtract 4}} \xleftarrow{x}$$

so

$$f^{-1} : x \mapsto \frac{x-4}{3}\,.$$

Click on the green square to return □

Exercise 4(b)
For the function $f : x \mapsto 4(x-1)$ the flow diagram is

$$\xrightarrow{x} \boxed{\text{subtract } 1} \xrightarrow{x-1} \boxed{\text{multiply by } 4} \xrightarrow{4(x-1)}$$

The inverse is thus

$$\xleftarrow{\frac{x}{4}+1} \boxed{\text{add } 1} \xleftarrow{\frac{x}{4}} \boxed{\text{divide by } 4} \xleftarrow{x}$$

Thus

$$f^{-1} : x \mapsto \frac{x}{4} + 1 \,.$$

Click on the green square to return □

Exercise 4(c)
For the function $f : x \mapsto 3(2x + 5)$ the flow diagram is

$$\xrightarrow{x} \boxed{\text{multiply by 2}} \xrightarrow{2x} \boxed{\text{add 5}} \xrightarrow{2x+5} \boxed{\text{multiply by 3}} \xrightarrow{3(2x+5)}$$

The inverse has the flow diagram

$$\xleftarrow{\frac{1}{2}\left(\frac{x}{3}-5\right)} \boxed{\text{divide by 2}} \xleftarrow{\frac{x}{3}-5} \boxed{\text{subtract 5}} \xleftarrow{\frac{x}{3}} \boxed{\text{divide by 3}} \xleftarrow{x}$$

Thus
$$f^{-1} : x \mapsto \frac{1}{2}\left(\frac{x}{3} - 5\right)$$

Click on the green square to return □

Exercise 4(d)
For the functions $g : x \mapsto -8x + 3$ the flow diagram is

$$\xrightarrow{x} \boxed{\text{multiply by } -8} \xrightarrow{-8x} \boxed{\text{add } 3} \xrightarrow{-8x+3}$$

The inverse function has flow diagram

$$\xrightarrow{-\frac{1}{8}(x-3)} \boxed{\text{divide by } -8} \xrightarrow{x-3} \boxed{\text{subtract } 3} \xrightarrow{x}$$

so the inverse function is

$$g^{-1} : x \mapsto -\frac{1}{8}(x - 3)\,.$$

Click on the green square to return

Exercise 4(e)

Before beginning this question the function may be simplified by noting that

$$\frac{1}{2}(3x+4)+6=\frac{1}{2}(3x+4)+\frac{12}{2}=\frac{1}{2}(3x+16)$$

so that the function is $g : x \mapsto \frac{1}{2}(3x+16)$. The flow diagram for this function is

$$\xrightarrow{x}\boxed{\text{multiply by 3}}\xrightarrow{3x}\boxed{\text{add 16}}\xrightarrow{3x+16}\boxed{\text{divide by 2}}\xrightarrow{\frac{1}{2}(3x+16)}$$

The inverse flow diagram is

$$\xleftarrow{\frac{1}{3}(2x-16)}\boxed{\text{divide by 3}}\xleftarrow{2x-16}\boxed{\text{subtract 16}}\xleftarrow{2x}\boxed{\text{multiply by 2}}\xleftarrow{x}$$

and $$g^{-1} : x \mapsto \frac{1}{3}(2x-16).$$

Click on the green square to return ☐

Exercise 4(f)

For the functions $g : x \mapsto \dfrac{2x+1}{3}$ the flow diagram is

$$\xrightarrow{x} \boxed{\text{multiply by 2}} \xrightarrow{2x} \boxed{\text{add 1}} \xrightarrow{2x+1} \boxed{\text{divide by 3}} \xrightarrow{\frac{1}{3}(2x+1)}$$

and the inverse is given by

$$\xleftarrow{\frac{1}{2}(3x-1)} \boxed{\text{divide by 2}} \xleftarrow{3x-1} \boxed{\text{subtract 1}} \xleftarrow{3x} \boxed{\text{multiply by 3}} \xleftarrow{x}$$

so that

$$g^{-1} : x \mapsto \frac{1}{2}(3x-1)\,.$$

Click on the green square to return ☐

Solutions to Quizzes

Solution to Quiz:

The flow diagram

$$\xrightarrow{x} \boxed{\text{cube}} \longrightarrow \boxed{\text{divide by 4}} \longrightarrow \boxed{\text{subtract 1}} \longrightarrow \boxed{\text{square}} \longrightarrow \boxed{\text{add 4}} \longrightarrow$$

is completed as

$$\xrightarrow{x} \boxed{\text{cube}} \xrightarrow{x^3} \boxed{\text{divide by 4}} \xrightarrow{\frac{x^3}{4}} \boxed{\text{subtract 1}} \xrightarrow{\frac{x^3}{4}-1}$$

$$\xrightarrow{\frac{x^3}{4}-1} \boxed{\text{square}} \xrightarrow{\left(\frac{x^3}{4}-1\right)^2} \boxed{\text{add 4}} \xrightarrow{\left(\frac{x^3}{4}-1\right)^2+4}$$

which is the function $k : x \mapsto \left(\dfrac{x^3}{4} - 1\right)^2 + 4$.

End Quiz

Solution to Quiz:
For the functions $f : x \mapsto 2x^2 - 1$ and $g : x \mapsto x - 3$, the composite function is $fg(x) = 2(x-3)^2 - 1$.

$$\begin{aligned}
2(x-3)^2 - 1 &= 7 \\
2(x-3)^2 &= 1 + 7 = 8 \\
(x-3)^2 &= 4 \\
x - 3 &= \pm 2 \\
x &= 3 \pm 2
\end{aligned}$$

so that $x = 5$ and $x = 1$ are both solutions.

End Quiz

Solution to Quiz:

The function $h = fg$ is $h : x \mapsto \dfrac{1}{2}(3x + 16)$. This is the function obtained by simplifying the function of exercise 4(e). The inverse of the function is thus

$$h^{-1} : x \mapsto \frac{1}{3}(2x - 16)\,,$$

as shown in that exercise.

End Quiz