Basic Mathematics

Indefinite Integration

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The aim of this package is to provide a short self assessment programme for students who want to be able to calculate basic indefinite integrals.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Anti-Derivatives

If $f = \frac{dF}{dx}$, we call F the anti-derivative (or indefinite integral) of f.

Example 1 If $f(x) = x$, we can find its anti-derivative by realising that for $F(x) = \frac{1}{2}x^2$

$$\frac{dF}{dx} = \frac{d}{dx}(\frac{1}{2}x^2) = \frac{1}{2} \times 2x = x = f(x)$$

Thus $F(x) = \frac{1}{2}x^2$ is an anti-derivative of $f(x) = x$.

However, if C is a constant:

$$\frac{d}{dx}(\frac{1}{2}x^2 + C) = \frac{1}{2} \times 2x = x$$

since the derivative of a constant is zero. The **general anti-derivative** of x is thus $\frac{1}{2}x^2 + C$ where C can be *any* constant.

Note that you should always check an anti-derivative F by differentiating it and seeing that you recover f.

Quiz Using $\frac{d(x^n)}{dx} = nx^{n-1}$, select an anti-derivative of x^6

(a) $6x^5$ (b) $\frac{1}{5}x^5$ (c) $\frac{1}{7}x^7$ (d) $\frac{1}{6}x^7$

In general the anti-derivative or integral of x^n is:

$$\text{If } f(x) = x^n \text{, then } F(x) = \frac{1}{n+1}x^{n+1} \quad \text{for } n \neq -1$$

N.B. this rule does not apply to $1/x = x^{-1}$. Since the derivative of $\ln(x)$ is $1/x$, the anti-derivative of $1/x$ is $\ln(x)$ – see later.

Also **note** that since $1 = x^0$, the rule says that the anti-derivative of 1 is x. This is correct since the derivative of x is 1.

We will now introduce two important properties of integrals, which follow from the corresponding rules for derivatives.

If a is any constant and $F(x)$ is the anti-derivative of $f(x)$, then

$$\frac{d}{dx}\left(aF(x)\right) = a\frac{d}{dx}F(x) = af(x)\,.$$

Thus

$aF(x)$ is the anti-derivative of $af(x)$

Quiz Use this property to select the general anti-derivative of $3x^{\frac{1}{2}}$ from the choices below.

(a) $2x^{\frac{3}{2}} + C$ (b) $\frac{3}{2}x^{-\frac{1}{2}} + C$ (c) $\frac{9}{2}x^{\frac{3}{2}} + C$ (d) $6\sqrt{x} + C$

If $\frac{dF}{dx} = f(x)$ and $\frac{dG}{dx} = g(x)$, from the sum rule of differentiation

$$\frac{d}{dx}(F+G) = \frac{d}{dx}F + \frac{d}{dx}G = f(x) + g(x)\,.$$

(See the package on the **product and quotient rules.**) This leads to the sum rule for integration:

> If $F(x)$ is the anti-derivative of $f(x)$ and $G(x)$ is the anti-derivative of $g(x)$, then $F(x)+G(x)$ is the anti-derivative of $f(x)+g(x)$.

Only one arbitrary constant C is needed in the anti-derivative of the sum of two (or more) functions.

Quiz Use this property to find the general anti-derivative of $3x^2 - 2x^3$.

(a) C (b) $x^3 - \frac{1}{2}x^4 + C$ (c) $\frac{3}{2}x^3 - \frac{2}{3}x^4 + C$ (d) $x^3 + \frac{2}{3}x + C$

We now introduce the integral notation to represent anti-derivatives.

2. Indefinite Integral Notation

The notation for an anti-derivative or indefinite integral is:

$$\text{if} \quad \frac{dF}{dx} = f(x)\,, \quad \text{then} \quad \int f(x)\,dx = F(x) + C$$

Here $\int$ is called the integral sign, while dx is called the measure and C is called the integration constant. We read this as "the integral of f of x with respect to x" or "the integral of f of x dx".

In other words $\int f(x)\,dx$ means the general anti-derivative of $f(x)$ *including* an integration constant.

Example 2 To calculate the integral $\int x^4\,dx$, we recall that the anti-derivative of x^n for $n \neq -1$ is $x^{n+1}/(n+1)$. Here $n = 4$, so we have

$$\int x^4\,dx = \frac{x^{4+1}}{4+1} + C = \frac{x^5}{5} + C$$

Quiz Select the correct result for the indefinite integral $\int \frac{1}{\sqrt{x}}\, dx$

(a) $-\frac{1}{2}x^{-\frac{3}{2}} + C$ (b) $2\sqrt{x} + C$ (c) $\frac{1}{2}x^{\frac{1}{2}} + C$ (d) $\frac{2}{\sqrt{x^2}} + C$

The previous rules for anti-derivatives may be expressed in integral notation as follows.

The integral of a function multiplied by any constant a is:

$$\int af(x)dx = a\int f(x)dx$$

The sum rule for integration states that:

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

To be able to integrate a greater number of functions, it is convenient first to recall the derivatives of some simple functions:

y	$\sin(ax)$	$\cos(ax)$	e^{ax}	$\ln(x)$
$\dfrac{dy}{dx}$	$a\cos(ax)$	$-a\sin(ax)$	$a\ e^{ax}$	$\dfrac{1}{x}$

Exercise 1. From the above table of derivatives calculate the indefinite integrals of the following functions: (click on the **green** letters for the solutions)

(a) $\sin(ax)\,,$ (b) $\cos(ax)\,,$

(c) $e^{ax}\,,$ (d) $\dfrac{1}{x}$

These results give the following table of indefinite integrals (the integration constants are omitted for reasons of space):

$y(x)$	x^n $(n \neq -1)$	$\sin(ax)$	$\cos(ax)$	e^{ax}	$\frac{1}{x}$
$\int y(x) dx$	$\frac{1}{n+1} x^{n+1}$	$-\frac{1}{a} \cos(ax)$	$\frac{1}{a} \sin(ax)$	$\frac{1}{a} e^{ax}$	$\ln(x)$

Exercise 2. From the above table, calculate the following integrals: (click on the **green** letters for the solutions)

(a) $\int x^7 \, dx$,

(b) $\int 2 \sin(3x) \, dx$,

(c) $\int 4 \cos(2x) \, dx$,

(d) $\int 15 \; e^{-5s} \, ds$,

(e) $\int \frac{3}{w} \, dw$,

(f) $\int (\; e^s + e^{-s}) \, ds$.

Quiz Select the indefinite integral of $4\sin(5x) + 5\cos(3x)$.

(a) $20\cos(5x) - 15\sin(3x) + C$
(b) $4\sin(\frac{5x^2}{2}) + 5\cos(\frac{3x^2}{2}) + C$
(c) $-\frac{2}{3}\cos(5x) + \frac{5}{4}\sin(3x) + C$
(d) $-\frac{4}{5}\cos(5x) + \frac{5}{3}\sin(3x) + C$

EXERCISE 3. It may be shown that

$$\frac{d}{dx}\left[x(\ln(x) - 1)\right] = \ln(x)\,.$$

(See the package on the **product and quotient rules** of differentiation.) From this result and the properties reviewed in the package on **logarithms** calculate the following integrals: (click on the **green** letters for the solutions)

(a) $\int \ln(x)\, dx\,,$
(b) $\int \ln(2x)\, dx\,,$
(c) $\int \ln(x^3)\, dx\,,$
(d) $\int \ln(3x^2)\, dx\,.$

Hint expressions like ln(2) are constants!

3. Fixing Integration Constants

Example 3 Consider a rocket whose velocity in metres per second at time t seconds after launch is $v = bt^2$ where $b = 3\,\mathrm{ms}^{-3}$. If at time $t = 2\,\mathrm{s}$ the rocket is at a position $x = 30\,\mathrm{m}$ away from the launch position, we can calculate its position at time $t\,\mathrm{s}$ as follows.

Velocity is the derivative of position with respect to time: $v = \dfrac{dx}{dt}$, so it follows that x is the integral of v $(= bt^2\,\mathrm{ms}^{-1})$:

$$x = \int 3t^2\,dt = 3 \times \frac{1}{3}t^3 + C = t^3 + C$$

The information that $x = 30\,\mathrm{m}$ at $t = 2\,\mathrm{s}$, can be substituted into the above equation to find the value of C:

$$\begin{aligned} 30 &= 2^3 + C \\ 30 &= 8 + C \\ \textit{i.e.,}\qquad 22 &= C\,. \end{aligned}$$

Thus at time $t\,\mathrm{s}$, the rocket is at $x = t^3 + 22\,\mathrm{m}$ from the launch site.

Quiz If $y = \int 3x\,dx$ and at $x = 2$, it is measured that $y = 4$, calculate the integration constant.

(a) $C = 2$ (b) $C = 4$ (c) $C = -2$ (d) $C = 10$

Quiz Find the position of an object at time $t = 4\,\mathrm{s}$ if its velocity is $v = \alpha t + \beta\,\mathrm{ms}^{-1}$ for $\alpha = 2\,\mathrm{ms}^{-2}$ and $\beta = 1\,\mathrm{ms}^{-1}$ and its position at $t = 1\,\mathrm{s}$ was $x = 2\,\mathrm{m}$.

(a) $12\,\mathrm{m}$ (b) $24\,\mathrm{m}$ (c) $0\,\mathrm{m}$ (d) $20\,\mathrm{m}$

Quiz Acceleration a is the rate of change of velocity v with respect to time t, i.e., $a = \dfrac{dv}{dt}$.

If a ball is thrown upwards on the Earth, its acceleration is constant and approximately $a = -10\,\mathrm{m\,s}^{-2}$. If its initial velocity was $3\,\mathrm{ms}^{-1}$, when does the ball stop moving upwards (i.e., at what time is its velocity zero)?

(a) $0.3\,\mathrm{s}$ (b) $1\,\mathrm{s}$ (c) $0.7\,\mathrm{s}$ (d) $0.5\,\mathrm{s}$

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. Which of the following is an anti-derivative with respect to x of $f(x) = 2\cos(3x)$?

(a) $2x\cos(3x)$ (b) $-6\sin(3x)$ (c) $\frac{2}{3}\sin(3x)$ (d) $\frac{2}{3}\sin(\frac{3}{2}x^2)$

2. What is the integral with respect to x of $f(x) = 11\exp(10x)$?

(a) $\frac{11}{10}\exp(10x) + C$ (b) $11\exp(5x^2) + C$

(c) $\exp(11x) + C$ (d) $110\exp(10x) + C$

3. If the speed of an object is given by $v = bt^{-\frac{1}{2}}\,\mathrm{ms}^{-1}$ for $b = 1\,\mathrm{ms}^{-\frac{1}{2}}$, what is its position x at time $t = 9\,\mathrm{s}$ if the object was at $x = 3\,\mathrm{m}$ at $t = 1\,\mathrm{s}$?

(a) $x = 7\,\mathrm{m}$ (b) $x = 11\,\mathrm{m}$ (c) $x = 4\,\mathrm{m}$ (d) $x = 0\,\mathrm{m}$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) To calculate the indefinite integral $\int \sin(ax)\,dx$ let us use the table of derivatives to find the function whose derivative is $\sin(ax)$.

From the table one can see that if $y = \cos(ax)$, then its derivative with respect to x is

$$\frac{d}{dx}(\cos(ax)) = -a\sin(ax), \quad \text{so} \quad \frac{d}{dx}\left(-\frac{1}{a}\ \cos(ax)\right) = \sin(ax)\,.$$

Thus one can conclude

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C\,.$$

Click on the **green** square to return

□

Exercise 1(b) We have to find the indefinite integral of $\cos(ax)$. From the table of derivatives we have

$$\frac{d}{dx}(\sin(ax)) = a\cos(ax), \quad \text{so} \quad \frac{d}{dx}\left(\frac{1}{a}\sin(ax)\right) = \cos(ax)\,.$$

This implies

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C\,.$$

Click on the **green** square to return

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Exercise 1(c) We have to find the integral of e^{ax}. From the table of derivatives

$$\frac{d}{dx}\left(e^{ax}\right) = a\ e^{ax}, \quad \text{so} \quad \frac{d}{dx}\left(\frac{1}{a}e^{ax}\right) = \ e^{ax}\,.$$

Thus the indefinite integral of e^{ax} is

$$\int e^{ax}\,dx = \frac{1}{a}\ e^{ax} + C\,.$$

Click on the **green** square to return □

Exercise 1(d) We need to find the function whose derivative is $\frac{1}{x}$. From the table of derivatives we see that the derivative of $\ln(x)$ with respect to x is

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}\,.$$

This implies that

$$\int \frac{1}{x}\,dx = \ln(x) + C\,.$$

Click on the **green** square to return □

Exercise 2(a) We want to calculate $\int x^7\,dx$. From the table of indefinite integrals, for any $n \neq -1$,

$$\int x^n dx = \frac{1}{n+1} x^{n+1}\,.$$

In the case of $n = 7(\neq -1)$,

$$\begin{aligned}\int x^7 dx &= \frac{1}{7+1} \times x^{7+1} + C \\ &= \frac{1}{8} x^8 + C\,.\end{aligned}$$

Checking this:

$$\frac{d}{dx}\left(\frac{1}{8}x^8 + C\right) = \frac{1}{8}\frac{d}{dx}\; x^8 = \frac{1}{8} \times 8\; x^7 = x^7\,.$$

Click on the **green** square to return

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Exercise 2(b) To calculate the integral $\int 2\sin(3x)\,dx$ we use the formula

$$\int \sin(ax)dx = -\frac{1}{a}\cos(ax)\,.$$

In our case $a = 3$. Thus we have

$$\begin{aligned}\int 2\sin(3x)dx &= 2\int \sin(3x)dx = 2\times\left(-\frac{1}{3}\cos(3x)\right) + C\\ &= -\frac{2}{3}\cos(3x) + C\,.\end{aligned}$$

Checking:

$$\frac{d}{dx}\left(-\frac{2}{3}\cos(3x) + C\right) = -\frac{2}{3}\frac{d}{dx}\cos(3x) = -\frac{2}{3}\times(-3\sin(3x)) = 2\sin(3x).$$

Click on the **green** square to return

□

Exercise 2(c) To calculate the integral $\int 4\cos(2x)\,dx$ we use the formula

$$\int \cos(ax)dx = \frac{1}{a}\sin(ax)\,,$$

with $a = 2$. This yields

$$\begin{aligned}\int 4\cos(2x)dx &= 4\int \cos(2x)dx \\ &= 4 \times \left(\frac{1}{2}\sin(2x)\right) \\ &= 2\sin(2x) + C\,.\end{aligned}$$

It may be checked that

$$\frac{d}{dx}\,(2\sin(2x) + C) = 2\frac{d}{dx}\,\sin(2x) = 2 \times (2\cos(2x)) = 4\cos(2x)\,.$$

Click on the **green** square to return

□

Exercise 2(d) To find the integral $\int 15\ e^{-5s}\, ds$ we use the formula

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

with $a = -5$. This gives

$$\begin{aligned} \int 15 e^{-5s} ds &= 15 \int e^{-5s} ds \\ &= 15 \times \left(-\frac{1}{5}\ e^{-5s} \right) \\ &= -3\ e^{-5s} + C\,, \end{aligned}$$

and indeed

$$\frac{d}{ds} \left(-3\ e^{-5s} + C \right) = -3 \frac{d}{ds} e^{-5s} = -3 \times \left(-5 e^{-5s} \right) = 15 e^{-5s}.$$

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$\square$

Exercise 2(e) To find the integral $\int \frac{3}{w}\,dw$ we use the formula

$$\int \frac{1}{x}\,dx = \ln(x)\,.$$

Thus we have

$$\begin{aligned}\int \frac{3}{w}\,dw = \int 3 \times \frac{1}{w}\,dw &= 3\int \frac{1}{w}\,dw \\ &= 3\ln(w) + C\,.\end{aligned}$$

This can be checked as follows

$$\frac{d}{dw}\left(3\ln(w) + C\right) = 3\frac{d}{dw}\ln(w) = 3 \times \frac{1}{w} = \frac{3}{w}\,.$$

Click on the **green** square to return □

Exercise 2(f) To find the integral $\int(e^s + e^{-s})\, ds$ we use the sum rule for integrals, rewriting it as the sum of two integrals

$$\int(e^s + e^{-s})\, ds = \int e^s\, ds + \int e^{-s}\, ds$$

and then use

$$\int e^{ax}\, dx = \frac{1}{a}\, e^{ax}.$$

Take $a = 1$ in the first integral and $a = -1$ in the second integral. This implies

$$\begin{aligned} \int(e^s + e^{-s})\, ds &= \int e^s\, ds + \int e^{-s}\, ds \\ &= e^s + \left(\frac{1}{-1}\right) e^{-s} + C \\ &= e^s - e^{-s} + C\,. \end{aligned}$$

Click on the **green** square to return □

Exercise 3(a) To calculate the indefinite integral $\int \ln(x)\,dx$ we have to find the function whose derivative is $\ln(x)$. We are given

$$\frac{d}{dx}\left[x(\ln(x)-1)\right] = \ln(x)\,.$$

This implies

$$\int \ln(x)\,dx \quad = \quad x\left[\ln(x)-1\right] + C\,.$$

This can be checked by differentiating $x\left[\ln(x)-1\right] + C$ using the product rule. (See the package on the **product and quotient rules** of differentiation.)

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Exercise 3(b) To calculate the indefinite integral $\int \ln(2x)\,dx$ we recall the following property of **logarithms**

$$\ln(ax) = \ln(a) + \ln(x)$$

and then use the integral $\int \ln(x)\,dx = x\,[\ln(x) - 1] + C$ calculated in Exercise 3(a). This gives

$$\begin{aligned}
\int \ln(2x)\,dx &= \int (\ln(2) + \ln(x))\ dx \\
&= \ln(2) \times \int 1\,dx + \int \ln(x)\,dx \\
&= x\ln(2) + x\,(\ln(x) - 1) + C \\
&= x\,(\ln(2) + \ln(x) - 1) + C \\
&= x\,(\ln(2x) - 1) + C\,.
\end{aligned}$$

In the last line we used $\ln(2) + \ln(x) = \ln(2x)$.

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Exercise 3(c) To calculate the indefinite integral $\int \ln(x^3)\, dx$ we first recall from the package on **logarithms** that

$$\ln(x^n) = n \ln(x)$$

and the integral

$$\int \ln(x)\, dx \quad = \quad x\,[\ln(x) - 1] + C$$

calculated in Exercise 3(a). This all gives

$$\begin{aligned} \int \ln(x^3)\, dx &= \int (3 \ln(x))\ dx \\ &= 3 \times \int \ln(x)\, dx \\ &= 3x\,(\ln(x) - 1) + C\,. \end{aligned}$$

Click on the **green** square to return □

Exercise 3(d) Using the rules from the package on **logarithms**, $\ln(3x^2)$ may be simplified

$$\ln(3x^2) = \ln(3) + \ln(x^2) = \ln(3) + 2\ln(x)\,.$$

Thus

$$\begin{aligned}
\int \ln(3x^2)\,dx &= \int (\ln(3) + 2\ln(x))\ dx \\
&= \ln(3) \times \int 1\,dx + 2 \times \int \ln(x)\,dx \\
&= \ln(3)x + 2x\,[\ln(x) - 1] + C \\
&= x\,[\ln(3) + 2\ln(x) - 2] + C \\
&= x\left[\ln(3x^2) - 2\right] + C\,,
\end{aligned}$$

where the final expression for $\ln(3x^2)$ is obtained by using the rules of logarithms.

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Solutions to Quizzes

Solution to Quiz: To find an anti-derivative of x^6 first calculate the derivative of $F(x) = \frac{1}{7}x^7$. Using the basic formula

$$\frac{d}{dx}x^n = nx^{n-1}$$

with $n = 7$

$$\begin{aligned} \frac{dF}{dx} &= \frac{d}{dx}\left(\frac{1}{7}x^7\right) && (1) \\ &= \frac{1}{7}\frac{d}{dx}\left(x^7\right) && (2) \\ &= \frac{1}{7} \times 7\, x^{7-1} && (3) \\ &= x^6\,. && (4) \end{aligned}$$

This result shows that the function $F(x) = \frac{1}{7}x^7 + C$ is the general anti-derivative of $f(x) = x^6$. End Quiz

Solution to Quiz: To find the general anti-derivative of $3x^{\frac{1}{2}}$, recall that for constant a the anti-derivative of $af(x)$ is $aF(x)$, where $F(x)$ is the anti-derivative of $f(x)$.

Thus the anti-derivative of $3x^{\frac{1}{2}}$ is $3 \times \left(\text{the anti-derivative of } x^{\frac{1}{2}}\right)$.

To calculate the anti-derivative of $x^{\frac{1}{2}}$ we recall the anti-derivative of $f(x) = x^n$ is $F(x) = \frac{1}{n+1}x^{n+1}$ for $n \neq -1$. In our case $n = \frac{1}{2}$ and therefore this result can be used. The anti-derivative of $x^{\frac{1}{2}}$ is thus

$$\frac{1}{\frac{1}{2}+1}\ x^{(\frac{1}{2}+1)} = \frac{1}{3/2}x^{3/2} = 1 \times \frac{2}{3}\ x^{\frac{3}{2}} = \frac{2}{3}x^{\frac{3}{2}}\ .$$

Thus the general anti-derivative of $3x^{\frac{1}{2}}$ is $3 \times \frac{2}{3}x^{\frac{3}{2}} + C = 2x^{\frac{3}{2}} + C$.

This result may be checked by differentiating $F(x) = 2x^{3/2} + C$.

End Quiz

Solution to Quiz: To find the general anti-derivative of $3x^2-2x^3$, we use the sum rule for anti-derivatives. The anti-derivative of $3x^2-2x^3$ is $\big(\text{anti-derivative of } 3x^2\big) - \big(\text{anti-derivative of } 2x^3\big)$. Since the anti-derivative of $f(x)=x^n$ is $F(x)=\frac{1}{n+1}x^{n+1}$ for $n \neq -1$, for $n=2$:

$$\text{anti-derivative of } x^2 = \frac{1}{2+1}x^{2+1} = \frac{1}{3}x^3\,.$$

Thus the anti-derivative of $3x^2$ is

$$3 \times \big(\text{anti-derivative of } x^2\big) = 3 \times \frac{1}{3}x^3 = x^3.$$

Similarly the anti-derivative of $2x^3$ is

$$2 \times \big(\text{anti-derivative of } x^3\big) = 2 \times \frac{1}{3+1}x^{3+1} = \frac{1}{2}x^4.$$

Putting these results together we find that the general anti-derivative of $3x^2-2x^3$ is

$$F(x) = x^3 - \frac{1}{2}x^4 + C\,,$$

which may be confirmed by differentiation. End Quiz

Solution to Quiz: To calculate the indefinite integral

$$\int \frac{1}{\sqrt{x}}\,dx = \int \frac{1}{x^{1/2}}\,dx = \int x^{-1/2}\,dx$$

we recall the basic result, that the anti-derivative of $f(x) = x^n$ is $F(x) = \frac{1}{n+1}x^{n+1}$ for $n \neq -1$. In this case $n = -\frac{1}{2}$ and so

$$\begin{aligned}\int x^{-1/2}\,dx &= \frac{1}{-\frac{1}{2}+1}\,x^{(-\frac{1}{2}+1)} + C = \frac{1}{\frac{1}{2}}x^{\frac{1}{2}} + C \\ &= 1 \times \frac{2}{1}x^{\frac{1}{2}} + C = 2x^{\frac{1}{2}} + C \\ &= 2\sqrt{x} + C\,,\end{aligned}$$

where we recall that dividing by a fraction is equivalent to multiplying by its inverse (see the package on **fractions**). End Quiz

Solution to Quiz: To evaluate $\int(4\sin(5x) + 5\cos(3x))\,dx$ we use the sum rule for indefinite integrals to rewrite the integral as the sum of two integrals. Using

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) \quad \text{and} \quad \int \cos(ax)\,dx = \frac{1}{a}\sin(ax)$$

we get

$$\begin{aligned}
\int(4\sin(5x) + 5\cos(3x))\,dx &= 4\int \sin(5x)\,dx + 5\int \cos(3x)\,dx \\
&= 4 \times (-\frac{1}{5})\,\cos(5x) + 5 \times \frac{1}{3}\sin(3x) + C \\
&= -\frac{4}{5}\,\cos(5x) + \frac{5}{3}\,\sin(3x) + C\,.
\end{aligned}$$

This can be checked by differentiation. End Quiz

Solution to Quiz: If $y = \int 3x\,dx$ and at $x = 2$, $y = 4$ then

$$\begin{aligned} y = \int 3x\,dx &= 3 \times \int x\,dx \\ &= 3 \times \frac{1}{2}x^{1+1} + C \\ &= \frac{3}{2}x^2 + C \end{aligned}$$

is the general solution. Substituting $x = 2$ and $y = 4$ into the above equation, the value of C is obtained

$$\begin{aligned} 4 &= \frac{3}{2} \times (2)^2 + C \\ 4 &= 6 + C \\ \text{i.e.,} \qquad C &= -2\,. \end{aligned}$$

Therefore, for all x, $y = \frac{3}{2}x^2 - 2$. End Quiz

Solution to Quiz:
We are told that $v = \alpha t + \beta$ with $\alpha = 2\text{ms}^{-2}$, $\beta = 1\text{ms}^{-1}$ and at $t = 1\text{s}$, $x = 2\text{m}$. Since x is the integral of v:

$$x = \int v\, dt = \int (2t+1)\, dt = 2 \times \int t\, dt + \int 1\, dt = t^2 + t + C\,.$$

The position at time $t = 1\,\text{s}$ was $x = 2\,\text{m}$ so these values may be substituted into the above equation to find C:

$$\begin{aligned} 2 &= 1^2 + 1 + C \\ 2 &= 2 + C \\ i.e., \qquad 0 &= C\,. \end{aligned}$$

Therefore, for all t, $x = t^2 + t + 0 = t^2 + t$. At $t = 4\,\text{s}$,

$$x = (4)^2 + 4 = 16 + 4 = 20 \text{ m}.$$

End Quiz

Solution to Quiz: We are given $a = \frac{dv}{dt} = -10\text{ms}^{-2}$ and initial velocity $v = 3\text{ms}^{-1}$, and want to find when the velocity is zero. Since $a = \frac{dv}{dt}$, velocity is the integral of acceleration, $v = \int a\,dt$. The acceleration of the ball is constant, $a = -10\text{ms}^{-2}$, so that

$$v = \int (-10)\ dt = -10 \times \int dt = -10t + C\,.$$

At $t = 0\,,\ v = 3\text{ms}^{-1}$, so these values may be substituted into the above equation to find the constant C:

$$\begin{aligned} 3 &= -10 \times 0 + C \\ 3 &= C\,. \end{aligned}$$

Thus $v = -10t + 3$ for this problem. Therefore if $v = 0$

$$\begin{aligned} 0 &= -10t + 3 \\ 10t &= 3\,, \qquad t = 3/10\,. \end{aligned}$$

The ball stops moving upwards at 0.3 s. End Quiz