Basic Mathematics

Log-Log Plots

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The aim of this package is to provide a short self assessment programme for students who wish to acquire an understanding of log-log plots.

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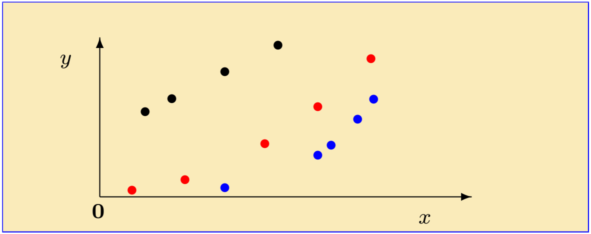
1. Introduction

Many quantities in science can be described by equations of the form, $y = Ax^n$. It is, though, not easy to distinguish between graphs of different power laws. Consider the data below:

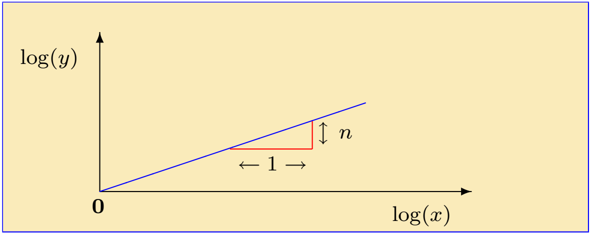


It is not easy to see that the red points lie on a quadratic ($y = Ax^2$) and that the blue data are on a quartic ($y = Ax^4$). It is, however, clear that the **black** points lie on a straight line! Results from the packages on **Logarithms** and **Straight Lines** enable us to recast the power curves as straight lines and so extract both n and A.

Example 1 Consider the equation $y = x^n$. This is a power curve, but if we take the logarithm of each side we obtain:

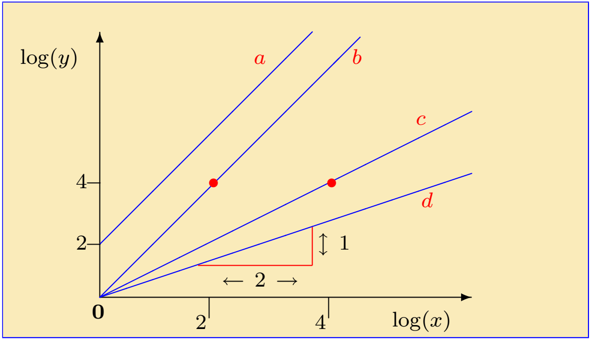
$$\begin{aligned} \log(y) &= \log(x^n) \\ &= n\log(x) \qquad \text{since} \quad \log(x^n) = n\log(x) \end{aligned}$$

If $Y = \log(y)$ and $X = \log(x)$ then $Y = nX$. This shows the linear relationship. Plotting Y against X, i.e., $\log(y)$ against $\log(x)$, leads to a straight line as shown below.



Here n is the slope of the line. Thus:
from a log-log plot, we can directly read off the power, n.

Quiz Which of the following lines is a log-log plot of $y = x^2$?



(a) a (b) b (c) c (d) d

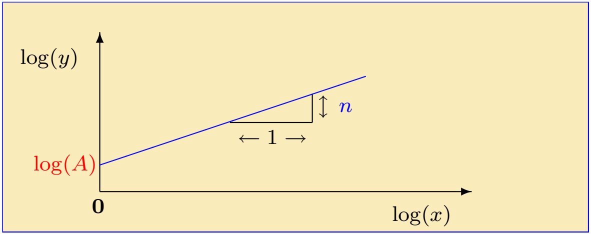
Note that the scales on the two axes are not the same.

2. Straight Lines from Curves

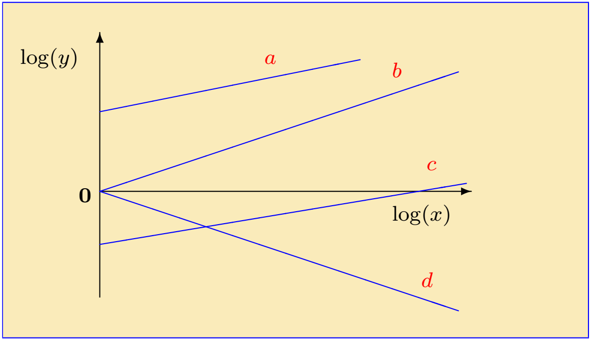
Example 2 Consider the more general equation $y = Ax^n$. Again we take the logarithm of each side:

$$\begin{aligned} \log(y) &= \log(Ax^n) \\ &= \log(A) + \log(x^n) \qquad \text{since} \quad \log(pq) = \log(p) + \log(q) \\ \therefore \log(y) &= n\log(x) + \log(A) \qquad \text{since} \quad \log(x^n) = n\log(x) \end{aligned}$$

The function $\log(y)$ is a linear function of $\log(x)$ and its graph is a straight line with gradient n which intercepts the $\log(y)$ axis at $\log(A)$.



Quiz Referring to the lines, a, b, c and d below, which of the following statements is **NOT** correct?



(a) If b corresponds to $y = x^3$, then d would describe $y = x^{-3}$.
(b) Lines a and c correspond to curves with the same power n.
(c) In the power law yielding c the coefficient A is negative.
(d) If b is from $y = x^3$, then in a the power n satisfies: $0 < n < 3$.

Exercise 1. Produce log-log plots for each of the following power curves. In each case give the gradient and the intercept on the $\log(y)$ axis. (Click on the **green** letters for the solutions).

(a) $y = x^{\frac{1}{3}}$
(b) $y = 10x^5$
(c) $y = 10x^{-2}$
(d) $y = \frac{1}{3}x^{-3}$

Quiz How does changing the base of the logarithm used (e.g., using $\ln(x)$ instead of $\log_{10}(x)$), change a log-log plot?

(a) The log-log plot is unchanged.
(b) Only the gradient changes.
(c) Only the intercept changes.
(d) Both the gradient and the intercept change.

Note that in an equation of the form $y = 5 + 3x^2$, taking logs directly does not help. This is because there is no rule to simplify $\log(5 + 3x^2)$. Instead we have to subtract the constant from each side.We then get: $y - 5 = 3x^2$, which leads to the straight line equation: $\log(y - 5) = 2\log(x) + \log(3)$.

3. Fitting Data

Suppose we want to see if some experimental data fits a power law of the form, $y = Ax^n$. We take logs of both sides and plot the points on a graph of $\log(y)$ against $\log(x)$. If they lie on a straight line (within experimental accuracy) then we conclude that y and x are related by a power law and the parameters A and n can be deduced from the graph. If the points do not lie on a straight line, then x and y are not related by an equation of this form.

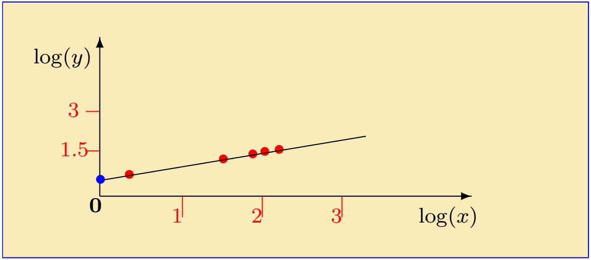
Example 3 Consider the following data:

x	2	30	70	100	150
y	4.24	16.4	25.1	30.0	36.7

To see if it obeys, $y = Ax^n$, we take logarithms of both sides. Here we use logarithms to the base 10. This gives the new table:

$\log_{10}(x)$	0.30	1.48	1.85	2	2.18
$\log_{10}(y)$	0.63	1.21	1.40	1.48	1.56

This is plotted on the next page.



It is evident that the red data points lie on a straight line. Therefore the original x and y values are related by a power law $y = Ax^n$.
To find the values of A and n, we first continue the line to the $\log_{10}(y)$ axis which it intercepts at the blue dot: $\log_{10}(A) = 0.48$. This means that $A = 10^{0.48} = 3.0$ (to 1 d.p.).
The gradient of the line is estimated using two of the points

$$n = \frac{\log(y_2) - \log(y_1)}{\log(x_2) - \log(x_1)} = \frac{1.56 - 0.63}{2.18 - 0.3} = 0.5 \quad \text{(to 1 d.p.)}$$

So the original data lies on the curve: $y = 3x^{\frac{1}{2}}$

Exercise 2. In the exercises below click on the **green** letters for the solutions.

(a) Rewrite the following expression in such a way that it gives the equation of a straight line

$$y = \sqrt{4x} + 2$$

(b) What is the difference between two power laws if, when they are plotted as a log-log graph, the gradients are the same, but the $\log(y)$ intercepts differ by $\log(3)$?

(c) Produce a log-log plot for the following data, show it obeys a power law and extract the law from the data.

x	5	15	30	50	95
y	10	90	360	1000	3610

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. The intercept and slope respectively of the log-log plot of $y = \frac{1}{2}x^2$

(a) $\frac{1}{2}$ & $\log(2)$ (b) $-\log(2)$ & 2

(c) $\log(2)$ & 2 (d) $\log(1/2)$ & $\log(2)$

2. If the $\log(y)$ axis intercept of the log-log plot of $y = Ax^n$ is negative, which of the following statements is true.

(a) $n < 0$ (b) $A = 1$ (c) $0 < A < 1$ (d) $n = -A$

3. The data below obeys a power law, $y = Ax^n$. Obtain the equation and select the correct statement.

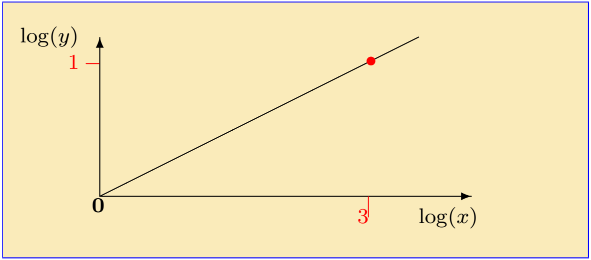
x	5	15	30	50	95
y	10	90	360	1000	3610

(a) $n = 3$ (b) $A = \frac{3}{2}$ (c) $n = 4$ (d) $A = \frac{1}{2}$

End Quiz Score: Correct

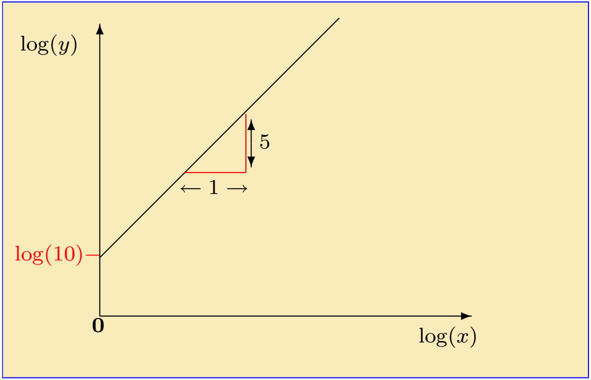
Solutions to Exercises

Exercise 1(a) For $y = x^{\frac{1}{3}}$, we get on taking logs: $\log(y) = \frac{1}{3}\log(x)$. This describes a line that passes through the origin and has slope $\frac{1}{3}$. It is sketched below:



Click on the **green** square to return □

Exercise 1(b) For $y = 10x^5$, we get on taking logarithms of each side: $\log(y) = 5\log(x) + \log(10)$. This describes a line that passes through $(0, \log(10))$ and has slope 5. It is sketched below:



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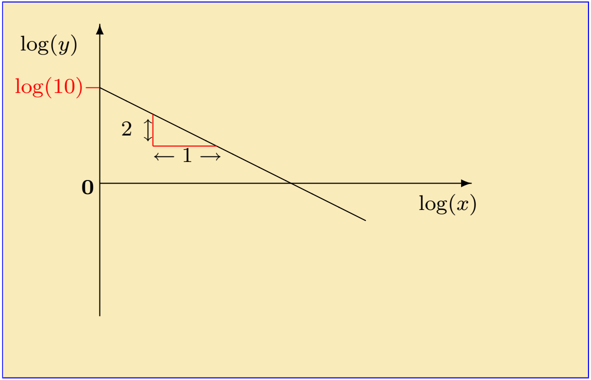
Exercise 1(c) The relation $y = 10x^{-2}$, can be re-expressed as $\log(y) = -2\log(x) + \log(10)$. This is sketched below.



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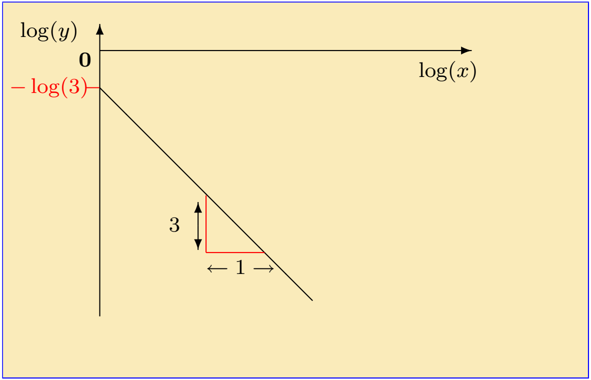
Exercise 1(d) If $y = \frac{1}{3}x^{-3}$, then $\log(y) = -3\log(x) + \log(\frac{1}{3})$. This can also be written as $\log(y) = -3\log(x) - \log(3)$. It is the equation of a line with slope -3 and intercept at $-\log(3)$. The line is sketched below.



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Exercise 2(a) $y = \sqrt{4x} + 4$ can be re-expressed as follows. Subtract 4 from each side

$$\begin{aligned} y - 4 &= \sqrt{4x} \\ y - 4 &= 2\sqrt{x} \\ y - 4 &= 2x^{\frac{1}{2}} \end{aligned}$$

Taking logarithms of each side yields

$$\log(y - 4) = \frac{1}{2}\log(x) + \log(2)$$

Thus plotting $\log(y-4)$ against $\log(x)$ would give a straight line with slope $\frac{1}{2}$ and intercept $\log(2)$ on the $\log(y-4)$ axis.

Click on the **green** square to return

□

Exercise 2(b) If $y = Ax^n$ then the log-log plot is the graph of the straight line

$$\log(y) = n\log(x) + \log(A)$$

So if the slope is the same the power n is the same in each case.

If the coefficients A_1 and A_2 differ by

$$\begin{aligned} \log(A_1) - \log(A_2) &= \log(3) \\ \text{then } \log\left(\frac{A_1}{A_2}\right) &= \log(3) \ \text{ since } \log(p/q) = \log(p) - \log(q) \end{aligned}$$

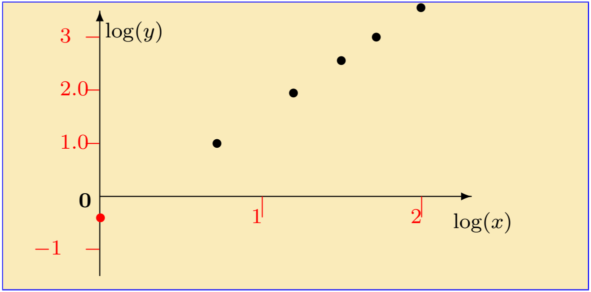
so it follows that the coefficients are related by $A_1 = 3A_2$.

Click on the **green** square to return

□

Exercise 2(c) To see if it obeys, $y = Ax^n$, we take logarithms to the base 10 of both sides. The table and graph are below:

$\log_{10}(x)$	0.70	1.18	1.48	1.70	1.98
$\log_{10}(y)$	1	1.95	2.56	3	3.56



The data points are fitted by a line that intercepts the $\log(y)$ axis at $\log(A) = -0.40$, so $A = 10^{-0.40} = 0.4$. The gradient can be calculated from $n = (3-1)/(1.70-0.70) = 2$. So the data lie on $y = 0.4x^2$. Click on the **green** square to return □

Solutions to Quizzes

Solution to Quiz: The curve is $y = x^2$. Taking logs of both sides gives: $\log(y) = \log(x^2) = 2\log(x)$, i.e., the log-log plot is a straight line through the origin with gradient 2.

Line b passes though the origin and through the point ($x = 2$, $y = 4$). From the package on **Straight Lines** we know that the gradient, m, of a straight line passing through ($\log(x_1)$, $\log(y_1)$) and ($\log(x_2)$, $\log(y_2)$) is given by

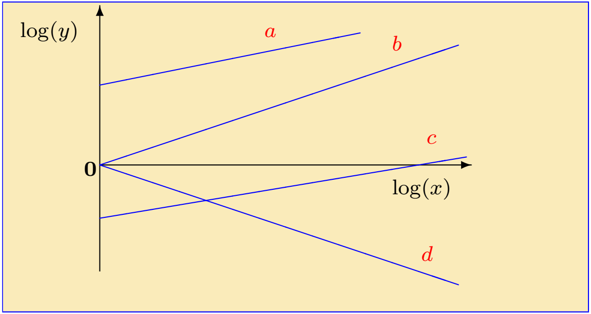
$$m = \frac{\log(y_2) - \log(y_1)}{\log(x_2) - \log(x_1)}$$

we see that the gradient of line b is given by

$$m_b = \frac{4 - 0}{2 - 0} = 2$$

This is therefore the correct log-log plot. End Quiz

Solution to Quiz:



If c corresponds to $y = Ax^n$, then $\log(y) = n\log(x) + \log(A)$. The intercept of line c on the $\log(y)$ axis is negative. This implies that $\log(A) < 0$, which means that $0 < A < 1$. It does **not** signify that A itself is negative. (Of course we also cannot take the logarithm of a negative number like this.)

It may be checked that the other statements are correct.

End Quiz

Solution to Quiz: The equation of a log-log plot is:

$$\log(y) = n\log(x) + \log(A)$$

If we change the base of the logarithm that is used, then the gradient n is unchanged but the intercept, $\log(A)$, is altered.

For example the log-log plot of $y = 3x^4$ in terms of logarithms to the base 10 is:

$$\log_{10}(y) = 4\log_{10}(x) + \log_{10}(3)$$

which has an intercept at $\log_{10}(3) = 0.477$ (to 3 d.p.) Using natural logarithms the equation would become

$$\ln(y) = 4\ln(x) + \ln(3)$$

This has the same gradient, but the intercept on the $\ln(y)$ axis is now at $\ln(3) = 1.099$ (to 3 d.p.)

The only exception to this is if $A = 1$, since $\log_N(1) = 0$ for all N.

End Quiz