Basic Mathematics

Quadratic Functions and their Graphs

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic competence at sketching graphs of quadratic functions.

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Table of Contents

1. Quadratic Functions (Introduction)

A general quadratic function has the form

$$y = ax^2 + bx + c\,,$$

where a, b, c are constants and $a \neq 0$. The simplest of these is

$$y = x^2$$

when $a = 1$ and $b = c = 0$. The following observations can be made about this simplest example.

- Since squaring any number gives a positive number, the values of y are all positive, except when $x = 0$, in which case $y = 0$.
- As x increases in size, so does x^2, but the increase in value is 'faster' than the increase in x.
- The graph of $y = x^2$ is symmetric about the y-axis ($x = 0$). For example, if $x = 3$ the corresponding y value is $3^2 = 9$. If $x = -3$, then the y value is $(-3)^2 = 9$. The two x values are equidistant from the y-axis, one to the left and one to the right, but the two y values are the same height above the x-axis.

This is sufficient to sketch the function.

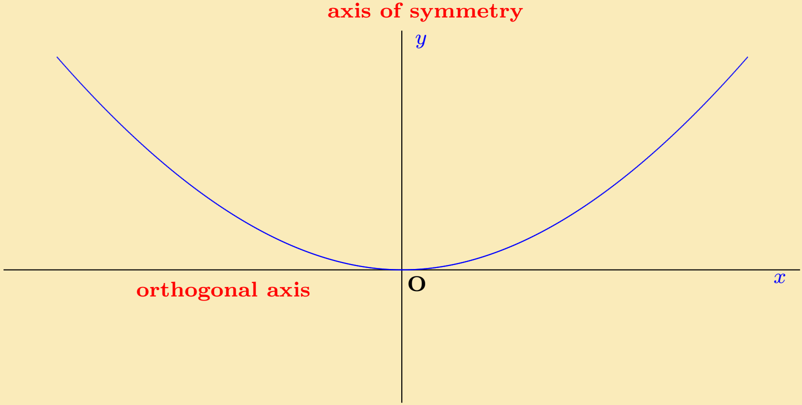


Diagram 1 Sketch of $y = x^2$.

Referring to diagram 1, the graph of $y = x^2$,

- the line $x = 0$ (i.e. the y-axis) will be called *the line of symmetry* for this quadratic.
- the line $y = 0$ (i.e. the x-axis) will be called *the orthogonal axis* for this quadratic.

If the equation is, say, $y = 2x^2$ then the graph will look similar to that of $y = x^2$ but will lie above it. For example, when $x = 1$ the value of x^2 is 1, the value of $2x^2$ is 2. The y value for $y = 2x^2$ is above that for $y = x^2$. Similarly, for the equation $y = x^2/2$, the graph looks similar to that of $y = x^2$ but now lies below it.

This is illustrated in the diagram on the next page.

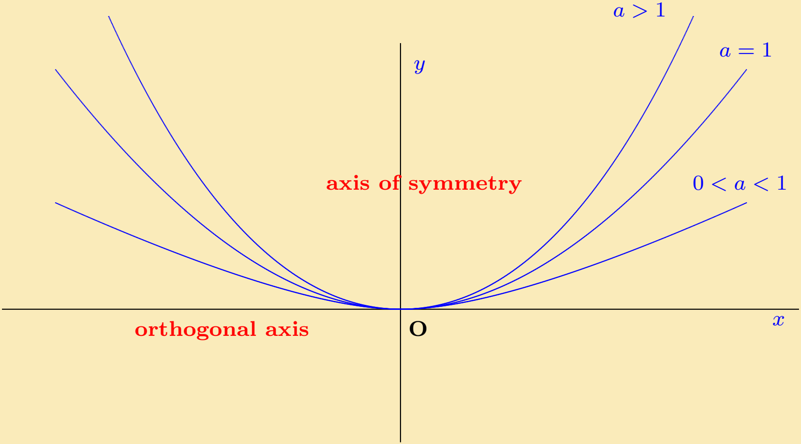


Diagram 2 Sketch of $y = ax^2$ for different *positive* values of a

The next possible choice is $a = -1$, with the equation $y = -x^2$. In this case the graph of the equation will have the same shape but now, instead of being *above* the x-axis it is *below*. When $x = 1$ the corresponding y value is -1. In a similar way, for differing negative values of a the graphs are below the x-axis.

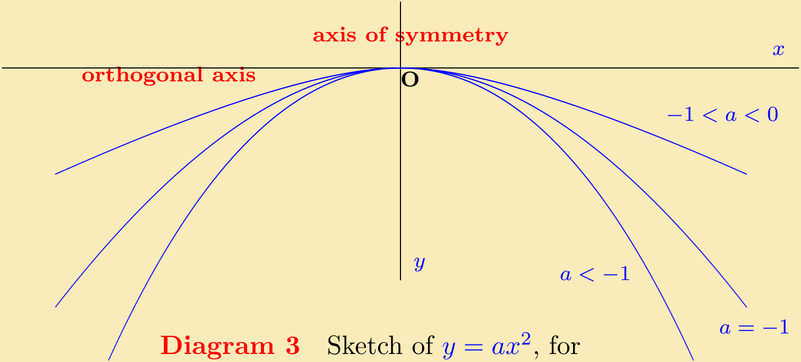
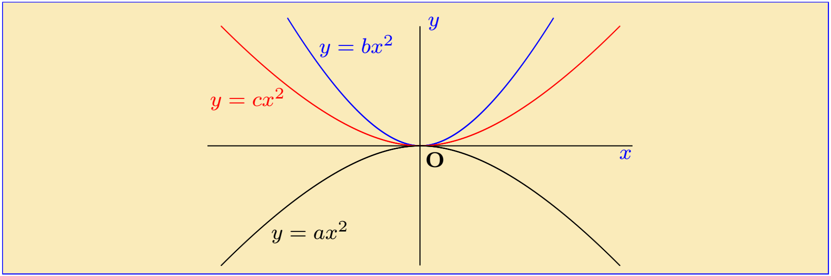


Diagram 3 Sketch of $y = ax^2$, for different *negative* values of a .

Quiz The diagram below shows a sketch of three quadratics.



Choose the appropriate option from the following.

(a) $a > b$ and $c > 0$,
(b) $b > c$ and $a > 0$,
(c) $c > b > a$,
(d) $b > c > a$.

2. Graph of $y = ax^2 + c$

This type of quadratic is similar to the basic ones of the previous pages but with a constant added, i.e. with the general form

$$y = ax^2 + c\,.$$

As a simple example of this take the case $y = x^2 + 2$. Comparing this with the function $y = x^2$, the only difference is the addition of 2 units.

- When $x = 1$, $x^2 = 1$, but $x^2 + 2 = 1 + 2 = 3$.
- When $x = 2$, $x^2 = 4$, but $x^2 + 2 = 4 + 2 = 6$.
- These points have been *lifted* by 2 units.
- This happens for *all* of the x values so the *shape* of the graph is unchanged but gets lifted by 2 units.

Similarly, the graph of $y = x^2 - 2$ will be *lowered* by 2 units.

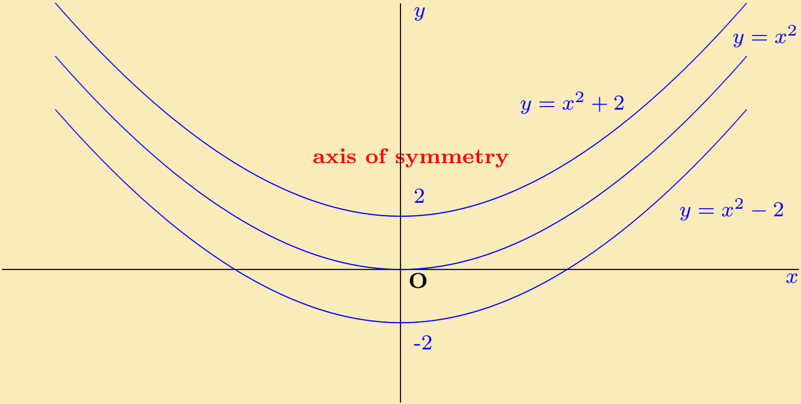


Diagram 4 Sketch of $y = x^2 + c$, for $c = 2, 0, -2$ respectively.

3. Graph of $y = a(x - k)^2$

In the examples considered so far, the *axis of symmetry* is the y-axis, i.e. the line $x = 0$. The next possibility is a quadratic which has its axis of symmetry *not on* the y-axis. An example of this is

$$y = (x - 3)^2,$$

which has the same shape and the same orthogonal axis as $y = x^2$ but with axis of symmetry the line $x = 3$.

- The points $x = 0$ and $x = 6$ are equidistant from 3.
- When $x = 0$ the y value is $(0 - 3)^2 = 9$.
- When $x = 6$ the y value is $(6 - 3)^2 = 9$.
- The points on the curve at these values are both 9 units above the x-axis.
- This is true for *all* numbers which are equidistant from 3.

The graph of $y = (x - 3)^2$ is illustrated on the next page.

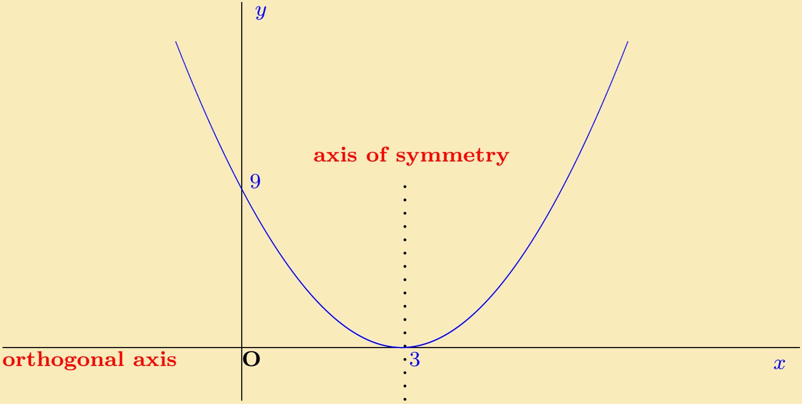


Diagram 5 Sketch of $y = (x - 3)^2$.
(Not to scale)

4. Graph of $y = a(x - k)^2 + m$

So far two separate cases have been discussed; first a standard quadratic has its *orthogonal axis* shifted up or down, second a standard quadratic has its *axis of symmetry* shifted left or right. The next step is to consider quadratics that incorporate both shifts.

Example 1 The quadratic $y = x^2$ is shifted so that its *axis of symmetry* is at $x = 3$ and its *orthogonal axis* is at $y = 2$.

(a) Write down the equation of the new curve.

(b) Find the coordinates of the point where it crosses the y axis.

(c) Sketch the curve.

Solution

(a) The new curve is symmetric about $x = 3$ and is shifted up by 2 units so its equation is $y = (x - 3)^2 + 2$.

(b) The curve crosses the y axis when $x = 0$. Putting this into the equation $y = (x - 3)^2 + 2$, the corresponding value of y is $y = (0 - 3)^2 + 2 = 11$, so the curve crosses the y axis at $y = 11$.

(c) The curve is sketched below.

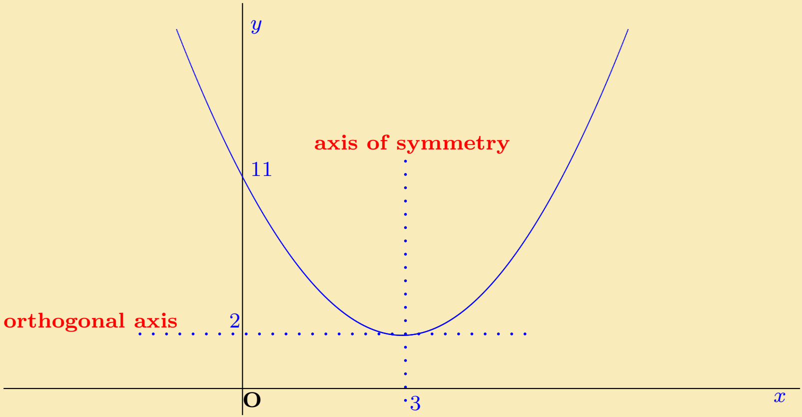


Diagram 5 Sketch of $y = (x-3)^2 + 2$.
(Not to scale)

EXERCISE 1. The curve $y = -2x^2$ is shifted so that its axis of symmetry is the line $x = -2$ and its orthogonal axis is $y = 8$. (Click on the green letters for solution.)

(a) Write down the equation of the new curve.

(b) Find the coordinates of the points where this new curve cuts the x and y axes.

(c) Sketch the curve.

EXERCISE 2. Repeat the above for each of the following.(Click on the green letters for solution.)

(a) The curve $y = x^2$ is shifted so that its axis of symmetry is the line $x = 7$ and its orthogonal axis is $y = 6$.

(b) The curve $y = x^2$ is shifted so that its axis of symmetry is the line $x = 7$ and its orthogonal axis is $y = -9$.

(c) The curve $y = -x^2$ is shifted so that its axis of symmetry is the line $x = 7$ and its orthogonal axis is $y = 9$.

5. Graph of a General Quadratic

The final section is about sketching general quadratic functions, i.e. ones of the form

$$y = ax^2 + bx + c\,.$$

The algebraic expression must be rearranged so that the *line of symmetry* and the *orthogonal axis* may be determined. The procedure required is *completing the square.* (See the package on **quadratics**.)

Example 2 A quadratic function is given as $y = -2x^2 + 4x + 16.$

(a) Complete the square on this function.

(b) Use this to determine the axis of symmetry and the orthogonal axis of the curve.

(c) Find the points on the x and y axes where the curve crosses them.

(d) Sketch the function.

Solution

(a) Completing the square:

$$\begin{aligned} y = -2x^2 + 4x + 16 &= -2(x^2 - 2x) + 16 \\ &= -2\left[(x-1)^2 - 1\right] + 16 \\ \text{i.e.}\quad y &= -2(x-1)^2 + 18 \end{aligned}$$

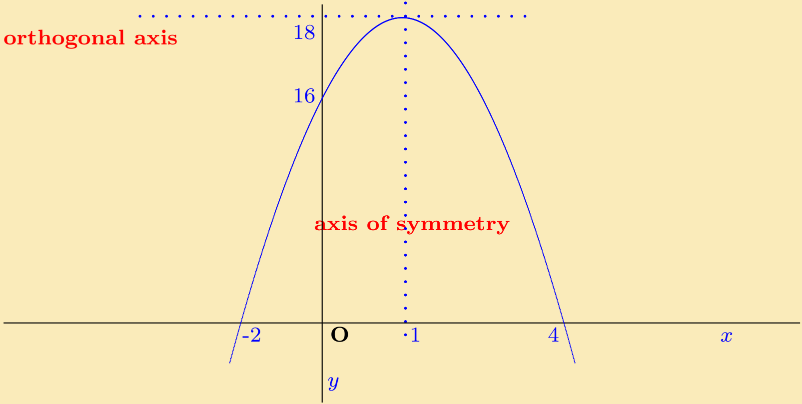
(b) This is the function $y = -2x^2$ moved so that its axis of symmetry is $x = 1$ and its orthogonal axis is $y = 18.$

(c) The function is $y = -2(x-1)^2 + 18$. This will cross the x-axis when $y = 0,$ i.e. when

$$\begin{aligned} -2(x-1)^2 + 18 &= 0 \\ 18 &= 2(x-1)^2 \\ 9 &= (x-1)^2 \\ \text{taking square roots}\quad x - 1 &= \pm 3 \\ x &= 1 \pm 3 \\ &= 4,\ \text{or}\ -2\,. \end{aligned}$$

Putting $x = 0$ into the original form of the function at the top of this page, gives $y = 16$, i.e. it crosses the y axis at $y = 16$.

(d) The function is sketched below.



Sketch of $y = -2x^2 + 4x + 16$.
(Not to scale)

Here are some exercises for practice.

Exercise 3. Use the method of example 2 to sketch each of the following quadratic functions. (Click on the green letters for solution.)

(a) $y = x^2 + 2x + 1$
(b) $y = 6 - x^2$
(c) $y = x^2 - 6x + 5$
(d) $4x - x^2$
(e) $y = x^2 + 2x + 5$
(f) $3 - 2x - x^2$

This section ends with a short quiz.

Quiz Which of the following pairs of lines is the axis of symmetry and orthogonal axis respectively of the quadratic function

$$y = -2x^2 - 8x?$$

(a) $x = 2$, $y = 8$,
(b) $x = 2$, $y = -8$,
(c) $x = -2$, $y = 8$,
(d) $x = -2$, $y = -8$.

6. Quiz on Quadratic Graphs

Begin Quiz Each of the following questions relates to the quadratic function $y = -x^2 + 6x + 7$.

1. At which of the following two points does it cross the x axis?
(a) $x = -1,\ 7$ (b) $x = 1,\ -7$ (c) $x = 1,\ 7$ (d) $x = -1,\ -7$

2. At which of the following does it cross the y axis?
(a) $y = 7$ (b) $y = 8$ (c) $y = 5$ (d) $y = 6$

3. Which Which of the following is the axis of symmetry?
(a) $x = 2$ (b)$x = -2$ (c) $x = -3$ (d) $x = 3$

4. Which of the following is the orthogonal axis?
(a) $y = 14$ (b)$y = 15$ (c) $y = 16$ (d) $y = 13$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) The equation is

$$y = -2(x+2)^2 + 8\,.$$

Click on the green square to return □

Exercise 1(b)

The curve cuts the y axis when $x = 0$. Putting $x = 0$ into the equation $y = -2(x+2)^2 + 8$, the corresponding y value is $-2(0+2)^2 + 8 = -2(2)^2 + 8 = -8 + 8 = 0$, i.e. $y = 0$.

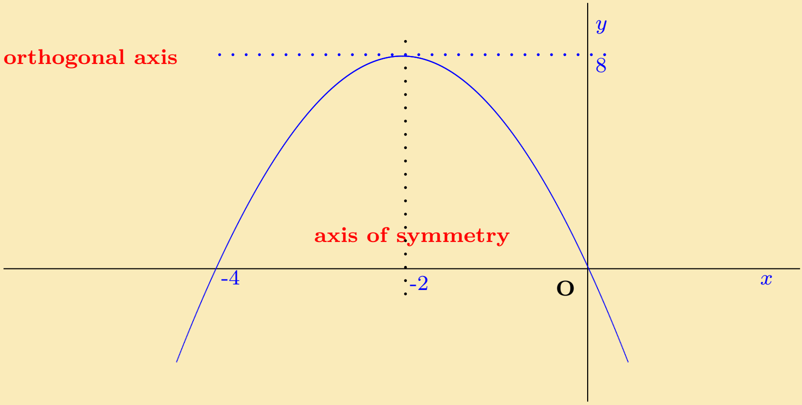
The curve cuts the x axis when $y = 0$. In this case putting the value $y = 0$ into the equation $y = -2(x+2)^2 + 8$ leads to

$$\begin{aligned} -2(x+2)^2 + 8 &= 0 \\ 8 &= 2(x+2)^2 \\ (x+2)^2 &= 4 \\ x+2 &= \pm 2 \\ x &= -2 \pm 2 \end{aligned}$$

so there are two solutions, $x = -4$ and $x = 0$.
To summarise the graph cuts the coordinate axes at the two points with coordinates $(-4, 0)$ and $(0, 0)$.

Click on the green square to return □

Exercise 1(c) The curve is sketched below.



Sketch of $y = -2(x+2)^2 + 8$.
(Not to scale)

Click on the green square to return

Exercise 2(a)

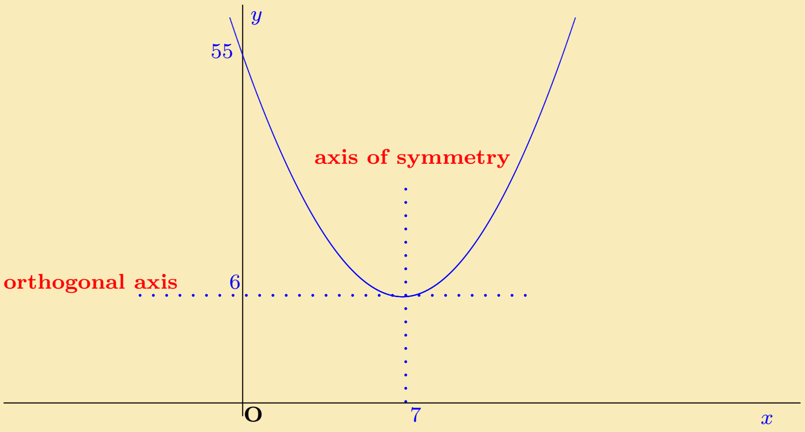
The equation of the shifted curve is

$$y = (x - 7)^2 + 6\,.$$

This will cross the y axis when $x = 0$, i.e. when

$$y = (0 - 7)^2 + 6 = (-7)^2 + 6 = 55\,.$$

It does not cross the x axis since its lowest point is on the orthogonal axis, which is $y = 6$. A sketch of this is on the next page.



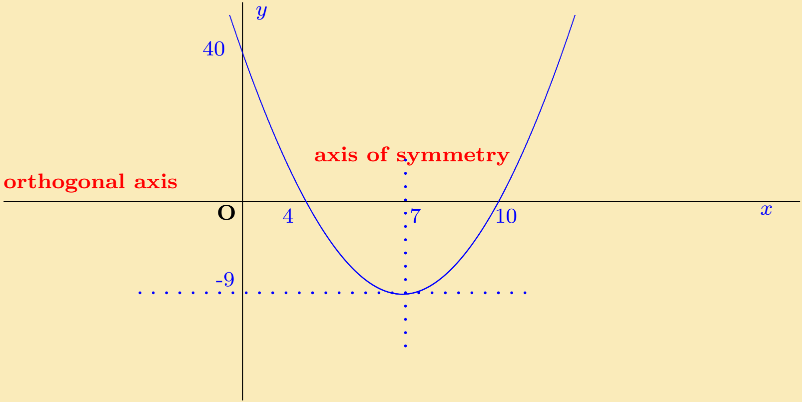
Sketch of $y = (x-7)^2 + 6$. (Not to scale)

Click on the green square to return

Exercise 2(b)
The curve will have the same shape as that in the previous part of this exercise but is now shifted *down* rather than up. The equation of the curve is $y = (x - 7)^2 - 9$. This will cross the y axis when $x = 0$ and $y = (0 - 7)^2 - 9 = 49 - 9 = 40$. It will cross the x axis when $y = 0$. Substituting this into the equation gives

$$\begin{aligned} (x-7)^2 - 9 &= 0 \\ (x-7)^2 &= 9 \\ x - 7 &= \pm 3 \\ x &= 7 \pm 3\,, \end{aligned}$$

i.e. the curve cuts the x axis at 4 and 10.
To summarise, the lowest point is on the *orthogonal axis* at $x = 7$, $y = -9$, it crosses the y axis at $y = 40$ and it crosses the x axis at $x = 4$, $x = 10$. The curve is sketched on the next page.



Sketch of $y = (x - 7)^2 - 9$.
(Not to scale)

Click on the green square to return □

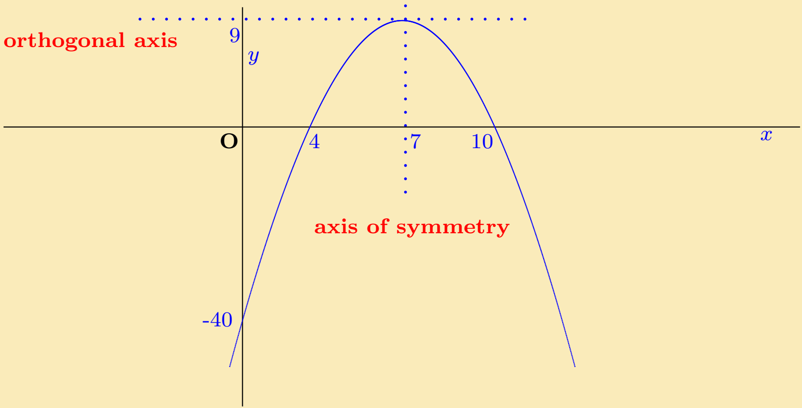
Exercise 2(c)
The equation for the new curve is

$$y = -(x-7)^2 + 9\,.$$

This will cross the y axis when $x = 0$, i.e. at $y = -(0-7)^2 + 9 = -49 + 9 = -40$. It crosses the x axis when $y = 0$, i.e.

$$\begin{aligned} -(x-7)^2 + 9 &= 0 \\ 9 &= (x-7)^2 \\ x - 7 &= \pm 3 \\ x &= 7 \pm 3\,, \end{aligned}$$

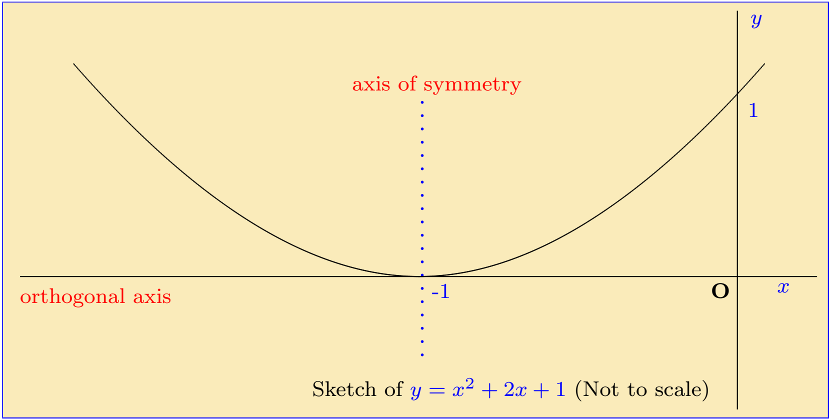
which gives $x = 4$ and $x = 10$.
To summarise, the curve has its highest point when $x = 7$ and $y = 9$, which is the orthogonal axis, it crosses the y axis at $y = -40$ and it crosses the x axis at $x = 4$ and $x = 10$. A sketch of this is on the next page.



Sketch of $y = -(x-7)^2 + 9$.
(Not to scale)

Click on the green square to return □

Exercise 3(a) The equation can be rewritten as $y = (x+1)^2$. A sketch of the function is shown below.



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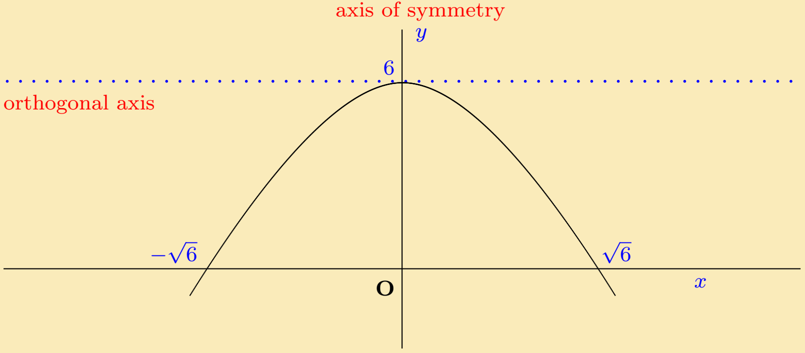
Exercise 3(b) The function $y = -x^2 + 6$ already has a complete square and is sketched below.



graph of $y = -x^2 + 6$ (Not to scale)

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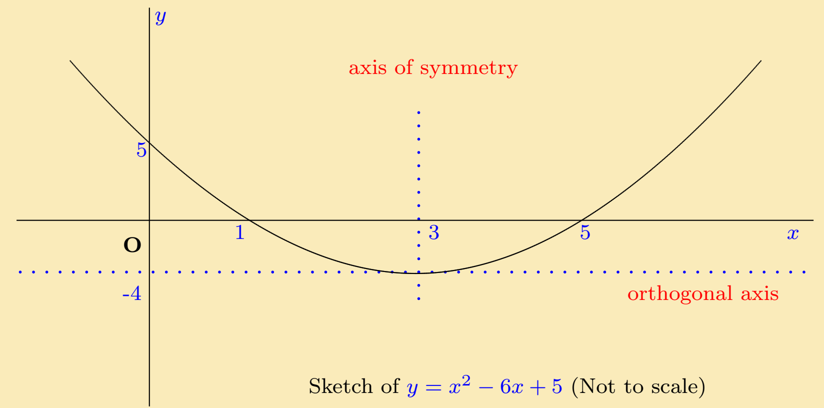
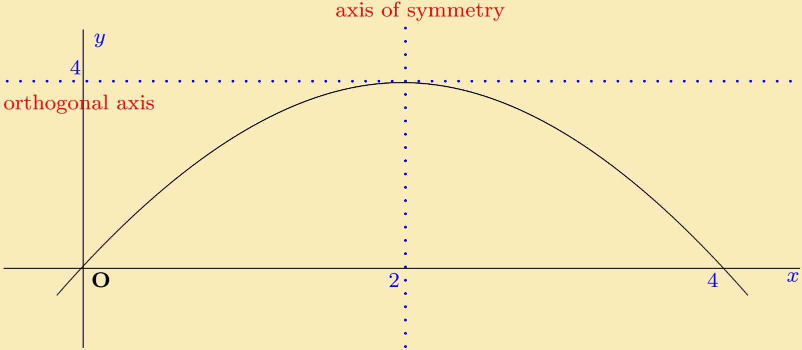
Exercise 3(c) On completing the square the original function $y = x^2 - 6x + 5$ becomes $y = (x-3)^2 - 4$.



Sketch of $y = x^2 - 6x + 5$ (Not to scale)

Click on the green square to return □

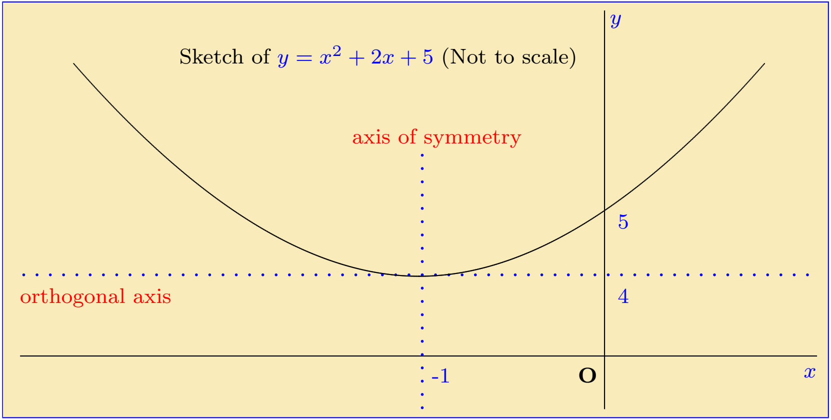
Exercise 3(d) On completing the square, this function becomes $y = -(x-2)^2 + 4$. The graph is as shown below.



Sketch of $y = 4x - x^2$ (Not to scale)

Click on the green square to return

Exercise 3(e) On completing the square the function becomes $y = (x+1)^2 + 4$. The graph is sketched below.



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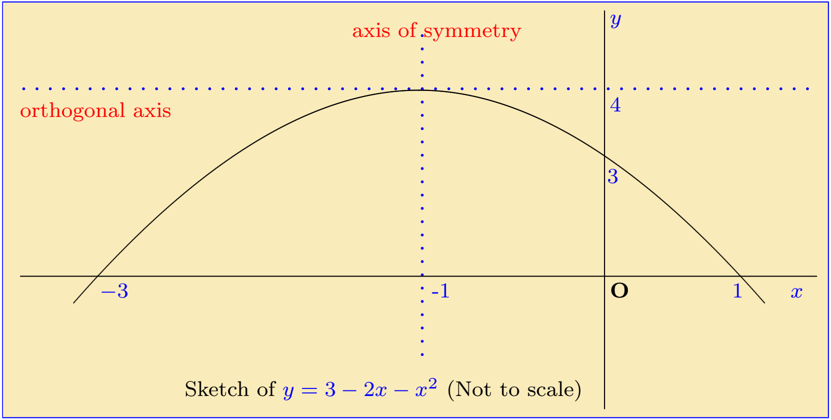
Exercise 3(f) On completing the square this function becomes $y = -(x+1)^2 + 4$. The sketch is shown below.



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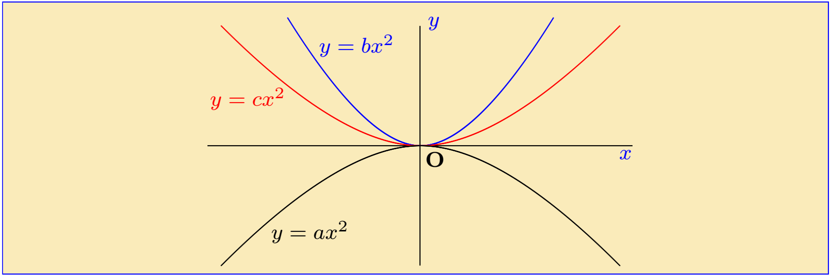
Solutions to Quizzes

Solution to Quiz:



The curves for $y = bx^2$ and $y = cx^2$ are both above the x axis and the former of these is above the latter, so $b > c$. The curve for $y = ax^2$ is below the x axis so $a < 0$. Since every positive number is greater than every negative number it follows that $b > c > a$.

End Quiz

Solution to Quiz:
Completing the square on $y = -2x^2 - 8x$ gives the function

$$y = -2(x + 2)^2 + 8\,,$$

i.e. the orthogonal axis is $y = 8$ and the axis of symmetry is $x = -2$. This is exactly the function which was examined in exercise 1 where the full details and sketch may be found. End Quiz