Basic Mathematics

Simultaneous Equations

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The aim of this package is to provide a short self assessment programme for students who are learning how to solve simultaneous equations.

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1. Two Equations and Two Unknowns

Many scientific problems lead to **simultaneous equations** containing quantities which need to be calculated. The simplest case is two simultaneous equations in two unknowns, say x and y.

Example 1 To start to see how we can solve such relations, consider

$$\begin{aligned} 4x + y &= 9 \\ 3x &= 6 \end{aligned}$$

There are two unknown variables x and y. However, the bottom equation only involves x and is solved by $x = 2$. We can then substitute this into the top equation to find

$$\begin{aligned} 4 \times 2 + y &= 9 \\ y &= 9 - 8 \\ y &= 1 \end{aligned}$$

The full solution is therefore $x = 2\,,\ y = 1\,.$

EXERCISE 1. Solve the following pairs of simultaneous equations (Click on the **green** letters for the solutions.)

(a)
$$\begin{aligned} x + y &= 3 \\ x &= 2 \end{aligned}$$

(b)
$$\begin{aligned} 4x - y &= 10 \\ y &= 2 \end{aligned}$$

(c)
$$\begin{aligned} z - x &= 2 \\ 2x &= -2 \end{aligned}$$

(d)
$$\begin{aligned} 3t + 2s &= 0 \\ s + 1 &= 2 \end{aligned}$$

Quiz What value of y solves the following pair of equations?

$$\begin{aligned} x + 2y &= 10 \\ x &= -2 \end{aligned}$$

(a) 12 (b) 4 (c) 8 (d) 6

2. Simultaneous Equations

More generally both equations may involve both unknowns.

Example 2 Consider

$$\begin{aligned} x + y &= 4 \qquad (1) \\ x - y &= 2 \qquad (2) \end{aligned}$$

Now add the left hand side of (1) to the left hand side of (2) and the right hand side of (1) to the right hand side of (2). The y's cancel and we get an equation for x alone

$$\begin{aligned} x + y + x - y &= 4 + 2 \\ 2x &= 6 \end{aligned}$$

which implies that $x = 3$. We can now insert this into (1) and so obtain:

$$3 + y = 4\,, \quad \Rightarrow \quad y = 4 - 3 = 1\,.$$

In other words the full solution is $x = 3\,,\ y = 1$

It is easy to check that you have the correct solution to simultaneous equations: by substituting your answers back into the original equations. We have already used (1) to find y, so let's check that (2) is correctly solved: we get $x - y = 3 - 1 = 2$ ✓

Always make such a check!

> **Example 2** illustrates the central idea of the method which is to combine the two equations so as to get a single equation for one variable and then use this to find the other unknown.

Exercise 2. Solve the following pairs of equations (Click on the **green** letters for the solutions.)

(a)
$$\begin{aligned} x + y &= 5 \\ x - y &= 1 \end{aligned}$$

(b)
$$\begin{aligned} 4x + 3y &= 7 \\ x - 3y &= -2 \end{aligned}$$

Example 3 Consider

$$\begin{aligned} x + 2y &= 4 \qquad (1)\\ x + y &= 3 \qquad (2) \end{aligned}$$

Subtracting these equations yields an equation in y, i.e., (1)-(2) gives

$$\begin{aligned} x + 2y - (x + y) &= 4 - 3 \\ y &= 1 \end{aligned}$$

Reinserting this result into (2) gives $x + 1 = 3$, so we obtain $x = 2$. **Check** the results by substituting them into (1)!

Quiz Solve the following simultaneous equations and select the correct result:

$$\begin{aligned} 3x + 3y &= 0 \\ 2x + 3y &= 1 \end{aligned}$$

(a) $x = 0, \quad y = 0$ (b) $x = -1, \quad y = 1$

(c) $x = 0, \quad y = 1$ (d) $x = 3, \quad y = 2$

3. A Systematic Approach

The first step in solving a system of two simultaneous equations is to eliminate one of the variables. This can be done by making the coefficient of x the same in each equation.

Example 4 Consider

$$\begin{aligned} 3x + 2y &= 4 \qquad (1) \\ 2x + y &= 3 \qquad (2) \end{aligned}$$

If we multiply (1) by 2, and (2) by 3, then we get

$$\begin{aligned} 6x + 4y &= 8 \\ 6x + 3y &= 9 \end{aligned}$$

We see that the coefficient of x is now the same in each equation! Subtracting them cancels ('eliminates') x and we can solve the simultaneous equations using the methods described above. Let us now work through an example.

Example 5 Consider the equations

$$\begin{aligned} 5x + 3y &= 7 \qquad (1) \\ 4x + 5y &= 3 \qquad (2) \end{aligned}$$

Multiply (1) by 4 (which is the coefficient of x in (2)) and also multiply (2) by 5(the coefficient of x in (1)).

$$\begin{aligned} 20x + 12y &= 28 \qquad (3) \\ 20x + 25y &= 15 \qquad (4) \end{aligned}$$

The coefficient of x is now the same in both equations. Subtracting (4)−(3) **eliminates** x:

$$25y - 12y = 15 - 28\,, \qquad \Rightarrow \quad 13y = -13$$

i.e., we have $y = -1$. Substituting this into (1) gives

$$5x - 3 = 7\,, \qquad \Rightarrow \quad 5x = 10$$

so that $x = 2$. Now check that $x = 2\,,\ y = -1$ by substitution into (2)!

Quiz To eliminate x from the following simultaneous equations, what should you multiply them by?

$$\begin{aligned} 3x - 2y &= 7 \\ 4x - 5y &= 7 \end{aligned}$$

(a) 7 & 7 (b) 4 & 3 (c) 3 & -2 (d) 3 & 4

Quiz To eliminate x from the simultaneous equations

$$\begin{aligned} 7x + 3y &= 13 \\ -2x + 5y &= 8 \end{aligned}$$

you can multiply (1) by -2 and (2) by 7. Which of the following equations for y will this procedure eventually yield?

(a) $29y = 82$ (b) $29y = 30$

(c) $41y = 82$ (d) $7y = -56$

EXERCISE 3. Solve the following equations by first eliminating x.

(a)
$$\begin{aligned} 3x + 4y &= 10 \\ 2x + 5y &= 9 \end{aligned}$$

(b)
$$\begin{aligned} 3x - 2y &= 9 \\ -x + 3y &= -3 \end{aligned}$$

(c)
$$\begin{aligned} 2x - y &= 5 \\ 3x + 4y &= 2 \end{aligned}$$

(d)
$$\begin{aligned} 5x + 7t &= 8 \\ 7x - 4t &= 25 \end{aligned}$$

Quiz Choose the solution of the following simultaneous equations

$$\begin{aligned} \frac{1}{2}x + 2y &= 3 \\ 2x + 3y &= 7 \end{aligned}$$

(a) $x = \frac{1}{2}\,, \qquad y = 2$

(b) $x = -\frac{1}{2}\,, \qquad y = 2$

(c) $x = 4\,, \qquad y = 0$

(d) $x = 2\,, \qquad y = 1$

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. If $x + y = 1$ and $x - y = 3$, what are x and y?

(a) $x = 2,\ y = -1$ (b) $x = -1,\ y = 2$
(c) $x = 2,\ y = 2$ (d) $x = 4,\ y = 1$

2. To eliminate x from the following equations, $ax + 2y = 4$ and $3x - 2ay = -17$, what do need to multiply them by?

(a) 4 & -17 (b) a & 3
(c) 3 & a (d) 2 & $-2a$

3. Solve $3x + 2y = 1$ and $2x + 3y = -1$.

(a) $x = 3,\ y = -4$ (b) $x = 5,\ y = 3$
(c) $x = -3,\ y = 5$ (d) $x = 1,\ y = -1$

4. For $2x - 3y = 1$ and $3x - 2y = 4$, find x and y.

(a) $x = 2,\ y = 1$ (b) $x = -1,\ y = -1$
(c) $x = 1,\ y = 2$ (d) $x = 3,\ y = -2$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) We have

$$\begin{aligned} x + y &= 3 \\ x &= 2 \end{aligned}$$

Substituting $x = 2$ into $x + y = 3$ we obtain:

$$\begin{aligned} 2 + y &= 3 \\ y &= 3 - 2 \\ \Rightarrow y &= 1 \end{aligned}$$

The solution is thus $x = 2\,,\ y = 1\,.$

Click on the **green** square to return □

Exercise 1(b) We have

$$\begin{aligned} 4x - y &= 10 \\ y &= 2 \end{aligned}$$

Substituting $y = 2$ into $4x - y = 10$ yields

$$\begin{aligned} 4x - 2 &= 10 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$

The solution is thus $x = 3\,,\ y = 2\,.$

Click on the **green** square to return $\square$

Exercise 1(c) We have

$$\begin{aligned} z - x &= 2 \\ 2x &= -2 \end{aligned}$$

From $2x = -2$ we have that $x = -1$. Inserting this into $z - x = 2$ we find

$$\begin{aligned} z - (-1) &= 2 \\ z + 1 &= 2 \\ z &= 1 \end{aligned}$$

The solution is thus $x = -1\,,\ z = 1\,.$
Click on the **green** square to return □

Exercise 1(d) We have

$$\begin{aligned} 3t + 2s &= 0 \\ s + 1 &= 2 \end{aligned}$$

From $s+1 = 2$, we have $s = 1$ and this can be inserted into $3t+2s = 0$ to give

$$\begin{aligned} 3t{+}2 &= 0 \\ 3t &= -2 \end{aligned}$$

$$t = -\frac{2}{3}$$

The solution is thus $s = 1\,,\ t = -\frac{2}{3}\,.$

Click on the **green** square to return

Exercise 2(a) We have the equations

$$\begin{aligned} x + y &= 5 \\ x - y &= 1 \end{aligned}$$

and adding them yields

$$2x = 6$$

so $x = 3$. This can now be inserted into the first equation to give

$$\begin{aligned} 3 + y &= 5 \\ y &= 2 \end{aligned}$$

The solution is thus $x = 3\,,\ y = 2\,.$

These results can be **checked** by inserting them into the second equation

$$x - y = 3 - 2 = 1\ \checkmark$$

Click on the **green** square to return □

Exercise 2(b) We have the equations

$$\begin{aligned} 4x + 3y &= 7 \\ x - 3y &= -2 \end{aligned}$$

and adding them yields

$$\begin{aligned} 4x + 3y + x - 3y &= 7-2 \\ 5x &= 5 \\ x &= 1 \end{aligned}$$

Substituting $x = 1$ into the first equation yields

$$\begin{aligned} 4 + 3y &= 7 \\ 3y &= 3 \\ y &= 1 \end{aligned}$$

This can now be checked by substitution into $x - 3y = 1 - 3 = -2$ ✓
Click on the **green** square to return □

Exercise 3(a) We have the equations

$$\begin{aligned} 3x + 4y &= 10 && (1) \\ 2x + 5y &= 9 && (2) \end{aligned}$$

and multiplying the first equation by 2 and the second by 3 yields:

$$\begin{aligned} 6x + 8y &= 20 && (3) \\ 6x + 15y &= 27 && (4) \end{aligned}$$

The coefficient of x is now the same and subtracting (3) from (4) yields an equation in y alone.

$$\begin{aligned} 15y - 8y &= 27 - 20 \\ 7y &= 7 \end{aligned}$$

so $y = 1$. Inserting this into (1) yields $3x + 4 = 10$, which implies that $3x = 6$ and so $x = 2$. Check $x = 2$, $y = 1$ by substitution into (2)!

Click on the **green** square to return □

Exercise 3(b) We have the equations

$$\begin{aligned} 3x - 2y &= 9 \qquad (1) \\ -x + 3y &= -3 \qquad (2) \end{aligned}$$

Multiplying the first equation by -1 and the second by 3 yields

$$\begin{aligned} -3x + 2y &= -9 \qquad (3) \\ -3x + 9y &= -9 \qquad (4) \end{aligned}$$

and subtracting (4) from (3) gives $-7y = 0$, so that $y = 0$.

Inserting this into (1) yields $x = 3$. The solution, $x = 3\,,\ y = 0$, should be checked by substitution into (2): $-x + 3y = -3 + 0$ ✓
Click on the **green** square to return □

Exercise 3(c) We have the equations

$$\begin{aligned} 2x - y &= 5 \quad (1) \\ 3x + 4y &= 2 \quad (2) \end{aligned}$$

Multiplying (1) by 3 and (2) by 2 yields

$$\begin{aligned} 6x - 3y &= 15 \quad (3) \\ 6x + 8y &= 4 \quad (4) \end{aligned}$$

and subtracting (4) from (3) gives $-11y = 11$, so $y = -1$.
Inserting this into the initial equation yields

$$\begin{aligned} 2x + 1 &= 5 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

Now check that $x = 2$, $y = -1$, by substitution into (2)!
Click on the **green** square to return □

Exercise 3(d) We have the equations

$$\begin{aligned} 5x + 7t &= 8 \qquad (1) \\ 7x - 4t &= 25 \qquad (2) \end{aligned}$$

Multiplying (1) by 7 and (2) by 5 yields

$$\begin{aligned} 35x + 49t &= 56 \qquad (3) \\ 35x - 20t &= 125 \qquad (4) \end{aligned}$$

and subtracting (4) from (3) gives

$$\begin{aligned} 49t + 20t &= 56 - 125 \\ 69t &= -69 \end{aligned}$$

So $t = -1$. Inserting this into (1) yields

$$\begin{aligned} 5x - 7 &= 8 \\ 5x &= 15 \end{aligned}$$

so we get $x = 3$, $t = -1$. Check this by substitution into (2)!
Click on the **green** square to return □

Solutions to Quizzes

Solution to Quiz: We are given

$$\begin{aligned} x + 2y &= 10 \\ x &= -2 \end{aligned}$$

Substituting $x = -2$ into $x + 2y = 10$ yields

$$\begin{aligned} -2 + 2y &= 10 \\ 2y &= 12 \\ y &= 6 \end{aligned}$$

The solution is thus $x = -2\,,\ y = 6\,.$ End Quiz

Solution to Quiz: We are given

$$\begin{aligned} 3x + 3y &= 0 \qquad (1) \\ 2x + 3y &= 1 \qquad (2) \end{aligned}$$

Subtracting these equations yields

$$\begin{aligned} 3x + 3y - (2x + 3y) &= 0-1 \\ x &= -1 \end{aligned}$$

This can now be substituted into (1) to yield

$$\begin{aligned} -3 + 3y &= 0 \\ 3y &= 3 \\ y &= 1 \end{aligned}$$

Check the solution, $x = -1$, $y = 1$, by substitution into (2).

End Quiz

Solution to Quiz: We have the equations

$$3x - 2y = 7 \tag{1}$$

$$4x - 5y = 7 \tag{2}$$

To eliminate x we have to multiply (1) by 4 and (2) by 3.

This procedure yields:

$$12x - 8y = 28 \tag{3}$$

$$12x - 15y = 21 \tag{4}$$

The x coefficient is then the same in each equation and so subtracting (4) from (3) indeed eliminates x. End Quiz

Solution to Quiz: We have

$$\begin{aligned} 7x + 3y &= 13 && (1) \\ -2x + 5y &= 8 && (2) \end{aligned}$$

Multiplication by -2 and 7 respectively yields

$$\begin{aligned} -14x - 6y &= -26 && (3) \\ -14x + 35y &= 56 && (4) \end{aligned}$$

Subtracting (4) from (3) cancels the x's and yields

$$\begin{aligned} -6y - 35y &= -26 - 56 \\ -41y &= -82 \\ 41y &= 82 \end{aligned}$$

This implies that $y = 2$ and on substitution into (1) we obtain $x = 1$. These answers can then be checked by substituting into (2).

End Quiz

Solution to Quiz: We have the equations

$$\frac{1}{2}x + 2y = 3 \tag{1}$$
$$2x + 3y = 7 \tag{2}$$

It is easiest here to multiply (1) by 4 and then subtract (2) from it. In this way we do not have unnecessary fractions. We find:

$$2x + 8y = 12 \tag{3}$$
$$2x + 3y = 7 \tag{4}$$

Subtracting them cancels the x's and yields

$$\begin{aligned} 5y &= 5 \\ y &= 1 \end{aligned}$$

Substituting this into (3) yields $x = 2$. The solution, $x = 2$, $y = 1$, can be checked by substitution into (2).

End Quiz