Basic Mathematics

# Units

**R Horan & M Lavelle**

The aim of this package is to provide a short self assessment programme for students who want to learn how to use and apply units in science.

Copyright © 2004 rhoran@plymouth.ac.uk , mlavelle@plymouth.ac.uk

Last Revision Date: October 16, 2005 Version 1.1

# Table of Contents



The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

# 1. SI Units

The ability to use units correctly is essential for scientists and engineers. Mistakes can be expensive:

Quiz What caused NASA's Mars Climate Orbiter probe to crash?

(a) It was hit by a meteorite
(b) Aliens
(c) A mix up over units
(d) Metal fatigue in a booster

You should **always state the units that you are using.**

Scientists and engineers use SI (Système International) units. Some of the basic units, which we will use in this package, are chosen to be:

| Quantity | Unit | Symbol |
|---|---|---|
| Length | Metre | m |
| Time | Second | s |
| Mass | Kilogram | kg |

Note that the kilogram (rather than the gram) is the SI unit of mass.

Many other units are derived from these base units. For example, the unit of area is square metres, *metres times metres*, which we write as $\mathrm{m}^2$. Similarly the unit of volume is cubic metres, *metres times metres times metres*, written as $\mathrm{m}^3$. (This follows the rules discussed in the package on **Powers**.)

**Example 1** What are the SI units of speed?

Speed is the distance travelled per unit of time:

$$\begin{aligned} \text{Speed} &= \frac{\text{distance (m)}}{\text{time (s)}} \\ \Rightarrow \text{Units of speed} &= \mathrm{m\,s^{-1}} \end{aligned}$$

This is read as metres per second.

**Quiz** What are the SI units for the density (mass per unit volume) of any substance ?

(a) $\mathrm{kg\,m^3}$ (b) $\mathrm{kg\,m^{-3}}$ (c) $\mathrm{kg^3m}$ (d) $\mathrm{kg\,m^{-1}}$

**Note:** the only case where you do not need to give units is for dimensionless ratios: e.g., the ratio of two distances or the ratio of two masses. These are independent of the units used.

**Example 2** A car atlas gives the distance from Plymouth to Lands End as 143 km or 89 miles. The distance from Plymouth to John o' Groats is stated to be 1271 km or 790 miles. The ratio of the two distances is

$$\begin{aligned} \frac{\text{Plymouth to John o' Groats}}{\text{Plymouth to Lands End}} &= \frac{1271\,\text{km}}{143\,\text{km}} = 8.9 \\ \textit{or} \qquad\qquad &= \frac{790\,\text{miles}}{89\,\text{miles}} = 8.9 \end{aligned}$$

It is nearly 9 times as far to John o' Groats as it is to Lands End no matter whether you use kilometres or miles to measure distances. (The ratio does not depend upon the units used.)

EXERCISE 1. Calculate the following in SI units (click on the **green** letters for the solutions).

(a) The area of a square with sides of length 1 m.

(b) The volume of a cube with sides of length 0.5 m.

(c) The average speed of a dog that runs 10 metres in 2 seconds.

(d) The density of a liquid whose volume is $0.2\,\mathrm{m}^3$ and whose mass is measured to be 0.8 kg.

Quiz The frequency of a wave is defined to be the number of cycles (or oscillations) per unit time. What are the SI units of frequency?

(a) s (b) $\mathrm{m\,s^{-1}}$ (c) $\mathrm{s^{-1}}$ (d) $\mathrm{s^{-2}}$

Quiz Acceleration is the rate of change of speed with time. What are the SI units of acceleration?

(a) $\mathrm{m\,s^{-2}}$ (b) $\mathrm{m\,s^{-1}}$ (c) $\mathrm{m^{-1}s^{-2}}$ (d) $\mathrm{m^{-2}s^{-2}}$

# 2. Derived Units

Some physical quantities are so widely used that their units have special names. They are called derived units because they are defined in terms of the base SI units.

**Example 3** The SI unit of force is the **Newton** (N). Let us see how it is constructed in terms of base units. Force is defined by

$$\begin{aligned} \text{Force} &= \text{mass} \times \text{acceleration} \\ \Rightarrow \text{Units of force} &= \mathrm{kg} \times \mathrm{m\,s^{-2}} = \mathrm{kg\,m\,s^{-2}} \end{aligned}$$

One Newton is short for one *kilogram metre per second squared*, i.e., it is the force needed to accelerate one kilogram by one metre per second squared.

Quiz The SI unit of energy is the **Joule** (J). From the expression for kinetic energy $E = \frac{1}{2}\mathrm{mv}^2$, where m is the mass and v is the speed of the object, express the Joule in terms of the base SI units.

(a) $\mathrm{kg\,m^2\,s^{-2}}$ (b) $\mathrm{kg^2\,m^{-2}\,s^{-2}}$ (c) $\mathrm{kg\,m\,s}$ (d) $\mathrm{kg^{-1}m\,s^{-2}}$

Exercise 2. Express the following in terms of base SI units (click on the **green** letters for the solutions).

(a) One **Pascal**, Pa, the unit of pressure, i.e., force per unit area

(b) One **Watt**, W, the unit of power, i.e., the rate of use of energy.

Exercise 3. Calculate the following in SI units (click on the **green** letters for the solutions).

(a) The frequency of a wave that oscillates 100 times in two seconds.

(b) The kinetic energy of a ball of mass 200 grams which moves through 6 metres in two seconds.

(c) The pressure on a surface of area 3 square metres which has a force of 60 Newtons applied to it.

(d) The power of an electrical device that uses 100 Joules of energy in one tenth of a second.

# 3. Scientific Notation

We encounter many large and small numbers in science and **scientific notation** (also called standard form) makes manipulating them easier. Write the number as one (non-zero) digit followed by the decimal places all multiplied by the appropriate power of ten.

**Example 4** Express the **astronomical unit** (AU), the mean distance from the Earth to the Sun, in scientific notation.

Since 1 AU $= 15,000,000,000,000\,\text{m}$, it may be written as

$$1.5\times10\times10\times10\times10\times10\times10\times10\times10\times10\times10\times10\times10\times10 = 1.5\times10^{13}\,\text{m}$$

**Example 5** A great advantage of scientific notation is that it is easy to multiply and divide numbers written in this way. E.g., if $a = 3\times10^6$ and $b = 2\times10^8$ then from the package on **Powers** we have

$$\begin{aligned} ab &= 3\times10^6\times2\times10^8 = 3\times2\times10^{6+8} = 6\times10^{14} \\ \frac{a}{b} &= \frac{3\times10^6}{2\times10^8} = \frac{3}{2}\times10^{6-8} = 1.5\times10^{-2} \end{aligned}$$

EXERCISE 4. Express the following in scientific notation (click on the **green** letters for the solutions).

(a) The radius of the earth (6,380,000 metres)

(b) Use this radius $r$ to estimate the volume of the Earth.

(c) The mass of the Earth is about $6 \times 10^{24}$ kg. Estimate the average density of the Earth.

(d) The frequency of a signal that oscillates 1,000,000 times in one hundredth of a second.

Quiz The frequency ($\nu$), wavelength ($\lambda$) and velocity of light ($c$) are related by $\nu = c/\lambda$. Calculate the frequency of X-rays of wavelength $6 \times 10^{-10}$ m. (The speed of light is $c = 3 \times 10^{8}\,\mathrm{m\,s^{-1}}$.)

(a) $1.8 \times 10^{-17}$ Hz

(b) $5.0 \times 10^{17}$ Hz

(c) $1.8 \times 10^{18}$ Hz

(d) $5.0 \times 10^{16}$ Hz

# 4. Prefixes

No single unit of length will be appropriate for describing, say, the distances between positions on a silicon chip and the separations of the planets in the solar system. To be able to do this we use **prefixes** (as in the familiar 1000 m=1 km, where 'k', which is short for kilo, means one thousand). Here are some of the more widespread prefixes:

| **Prefix** | **Factor** | **Symbol** |
|---|---|---|
| giga | $1,000,000,000 = 10^9$ | G |
| mega | $1,000,000 = 10^6$ | M |
| kilo | $1,000 = 10^3$ | k |
| milli | $0.001 = 10^{-3}$ | m |
| micro | $0.000,001 = 10^{-6}$ | $\mu$ |
| nano | $0.000,000,001 = 10^{-9}$ | n |

So there are one thousand grams in a kilogram and one thousandth of a second is a millisecond.

**Example 6** A surveyor measures the distance by road from Plymouth to Lands End to be 143,000 m. What is this in kilometres?

Since $1\,\mathrm{km} = 1,000\,\mathrm{m}$, we have

$$143,000\,\mathrm{m} = \frac{143,000}{1,000} = 143\,\mathrm{km}$$

Exercise 5. Express the following with the requested prefix (click on the **green** letters for the solutions).

(a) The length 12.7 nm (in metres)

(b) Air pressure: 101 kPa (in Pascal)

(c) The thrust of the first European moon mission Smart 1 launched in 2003 is 0.07 N. (in milli-Newtons).

(d) The power of the wind farm in Delabole, Cornwall: ten turbines each of power 400 kW (in MW).

Quiz What is the pressure exerted by a force of 50 N on a nailhead of area $0.002\,\mathrm{m}^2$?

(a) 25 kPa (b) 2,500 Pa (c) 4 kPa (d) $2.5\times10^{-3}$ MPa

# 5. Final Quiz

Begin Quiz Choose the solutions from the options given.

**1.** How are Joules and Newtons related?
(a) Not at all (b) $1\,\mathrm{J} = 1\,\mathrm{N\,s^{-1}}$
(c) $1\,\mathrm{J} = 1\,\mathrm{N\,m\,s^{-1}}$ (d) $1\,\mathrm{J} = 1\,\mathrm{N\,m}$

**2.** If a particle travels a distance of $3 \times 10^3\,\mathrm{m}$ in $t = 3 \times 10^{-3}\,\mathrm{s}$, calculate its speed.
(a) $9 \times 10^9\,\mathrm{m\,s^{-1}}$ (b) $1 \times 10^6\,\mathrm{m\,s}$
(c) $6 \times 10^{-6}\,\mathrm{m\,s^{-1}}$ (d) $1 \times 10^6\,\mathrm{m\,s^{-1}}$

**3.** Express the area of a square of side $5\,\mu\mathrm{m}$ in square metres.
(a) $25\,\mu\mathrm{m}^2$ (b) $2.5 \times 10^{-12}\,\mathrm{m}^2$
(c) $2.5 \times 10^{-11}\,\mathrm{m}^2$ (d) $2.5 \times 10^{-13}\,\mathrm{m}^2$

**4.** The energy $E$ of a photon with frequency $\nu$ is given by $E = h\nu$ where $h$ is Planck's constant. What are the units of $h$?
(a) $\mathrm{J\,s^{-1}}$ (b) $\mathrm{kg\,m\,s^{-1}}$ (c) $\mathrm{J^{-1}s}$ (d) $\mathrm{Js}$

End Quiz Score: Correct

# Solutions to Exercises

**Exercise 1(a)** The area of a square is the length of its side squared:

$$\begin{aligned} \text{Area} &= 1\,\text{m} \times 1\,\text{m} \\ &= 1\text{m}^2 \end{aligned}$$

The area is 1 square metre (or 1 metre squared).

Click on the **green** square to return

□

**Exercise 1(b)** The volume of a cube is given by the cube of the length of its side.

$$\begin{aligned} \text{Volume} &= 0.5\,\text{m} \times 0.5\,\text{m} \times 0.5\,\text{m} \\ &= 0.5^3\,\text{m}^3 \\ &= 0.125\,\text{m}^3 \end{aligned}$$

The volume is 0.125 cubic metres (or 0.125 metres cubed).

Click on the **green** square to return

□

**Exercise 1(c)** Speed is given by distance over time. So

$$\begin{aligned} \text{Speed} &= \frac{10\,\text{m}}{2\,\text{s}} \\ &= 5\,\text{m}\,\text{s}^{-1} \end{aligned}$$

So the average speed of the dog was 5 metres per second.

Click on the **green** square to return □

**Exercise 1(d)** Since the density is the mass per unit volume we have:

$$\text{Density} = \frac{0.8\,\text{kg}}{0.2\,\text{m}^3} = 4\,\text{kg}\,\text{m}^{-3}$$

So the density is 4 kilograms per metre cubed (or 4 kilograms per cubic metre).

Click on the **green** square to return □

**Exercise 2(a)** Pressure is force per unit area:

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}}$$

The SI units of force are $\text{kg}\,\text{m}\,\text{s}^{-2}$ and the units of area are $\text{m}^2$, so we obtain

$$\begin{aligned}\text{Units of pressure} &= \frac{\text{kg}\,\text{m}\,\text{s}^{-2}}{\text{m}^2} \\ &= \text{kg}\,\text{m}^{-1}\text{s}^{-2}\end{aligned}$$

One Pascal is thus one kilogram per metre per second squared.

Click on the **green** square to return □

**Exercise 2(b)** Power is the rate of use of energy:

$$\text{Power} = \frac{\text{Energy}}{\text{Time}}$$

The SI units of energy are $\text{kg}\,\text{m}^2\text{s}^{-2}$ and the units of time are s, so we obtain

$$\begin{aligned}\text{Units of power} &= \frac{\text{kg}\,\text{m}^2\text{s}^{-2}}{\text{s}} \\ &= \text{kg}\,\text{m}^2\text{s}^{-3}\end{aligned}$$

One Watt is thus one kilogram, metre squared per second cubed.

It is much easier to understand what this unit means if we recall that the derived unit of energy is the Joule. We thus simply say that a power of one Watt is one Joule per second. This shows the usefulness of derived units.

Click on the **green** square to return □

**Exercise 3(a)** If a wave oscillates 100 times in 2 seconds, the frequency is given by :

$$\begin{aligned}\text{Oscillations per second} &= \frac{100}{2\,\text{s}} = 50\,\text{s}^{-1} \\ \text{or} &= 50\,\text{Hz}\end{aligned}$$

Click on the **green** square to return

□

**Exercise 3(b)** The ball moves through 6 m in 2 s. But to calculate its kinetic energy, let us first express its mass of 200 g in SI units! This is 200/1000=0.2 kg. The average speed is

$$\begin{aligned} \text{Speed} &= \frac{\text{Distance}}{\text{Time}} \\ v &= \frac{6}{2} = 3\,\text{m}\,\text{s}^{-1} \end{aligned}$$

We now substitute this into the formula for kinetic energy: $E = \frac{1}{2}mv^2$. We get

$$E = 0.2\,\text{kg} \times (3\,\text{m}\,\text{s}^{-1})^2 = 0.2 \times 9\,\text{kg}\,\text{m}^2\text{s}^{-2} = 1.8\,\text{J}$$

where we recall that the Joule is the SI unit of energy.

Click on the **green** square to return

□

**Exercise 3(c)** Pressure is force over area. So if we apply 60 N to $3\,\mathrm{m}^2$, we obtain

$$\begin{aligned} \text{Pressure} &= \frac{60\,(\mathrm{N})}{3\,(\mathrm{m}^2)} \\ &= 20\,\mathrm{Pa} \end{aligned}$$

where we recall that a Pascal is a Newton per metre squared.

Click on the **green** square to return □

**Exercise 3(d)** If the device uses 100 Joules in one tenth of a second, then its power is:

$$\begin{aligned} \text{Power} &= \frac{100\,(\text{J})}{0.1\,\text{s}} \\ &= 1,000\text{W} \end{aligned}$$

where we recall that a Watt is a Joule per second.

Click on the **green** square to return □

**Exercise 4(a)** The radius, 6,380,000 metres, may be written more simply as:

$$\text{Radius of earth} = 6.38 \times 10^6 \,\text{m}\,.$$

Click on the **green** square to return

□

**Exercise 4(b)** To calculate its volume, we assume the Earth is roughly spherical with volume, $V = \frac{4}{3}\pi r^3$.

$$\begin{aligned} \text{Volume of Earth} &= \frac{4}{3}\pi(6.38 \times 10^6\,\text{m})^3 \\ &= \frac{4}{3}\pi \times 6.38^3 \times 10^{18}\,\text{m}^3 \\ &= 1.1 \times 10^{21}\,\text{m}^3 \end{aligned}$$

where we used $(10^a)^b = 10^{ab}$ and the answer is given to 2 significant figures.

Click on the **green** square to return □

**Exercise 4(c)** The density of the Earth is its mass over its volume:

$$\begin{aligned}\text{Density of Earth} &= \frac{6\times 10^{24}\,\text{kg}}{1.1\times 10^{21}\,\text{m}^3}\\ &\cong 5\times 10^3\,\text{kg}\,\text{m}^{-3}\end{aligned}$$

The density is therefore roughly 5,000 kilograms per cubic metres.

This can be compared with the density of water (roughly 1 kg per litre). Since 1 litre is $10\times 10\times 10 = 1,000\,\text{cm}^3$, which is $10^3/(100)^3 = 10^{-3}\,\text{m}^3$, we see that the density of water is roughly

$$\text{Density of water} = \frac{1(\text{kg})}{10^{-3}(\text{m}^3)} = 1,000\,\text{kg}\,\text{m}^{-3}$$

.

Click on the **green** square to return

**Exercise 4(d)** If the signal oscillates $10^6$ times in $10^{-2}$ s, its frequency is:

$$\begin{aligned} \text{Frequency} &= \frac{10^6}{10^{-2}\,\text{s}} \\ &= 10^{6-(-2)}\text{s}^{-1} \\ &= 10^8\,\text{Hz} \end{aligned}$$

where we recall that a Hertz is one cycle per second.

Click on the **green** square to return □

**Exercise 5(a)** To express 12.7 nm in metres, we first need to recall that 1 nm$= 1 \times 10^{-9}$m. Therefore

$$\begin{aligned} 12.7\text{nm} &= 12.7 \times 1 \times 10^{-9}\,\text{m} \\ &= 1.27 \times 10^{-8}\text{m} \end{aligned}$$

Click on the **green** square to return □

**Exercise 5(b)** Air pressure at sea level is roughly 101 kPa. Since 1 kPa=100 Pa,

$$\begin{aligned} 101\,\mathrm{kPa} &= 101 \times 1,000\,\mathrm{Pa} \\ &= 101,000\,\mathrm{Pa} \end{aligned}$$

This can also be written as $1.01 \times 10^5$ Pa. It is produced by roughly thirty kilometres of atmosphere pressing down on us!

Click on the **green** square to return □

**Exercise 5(c)** The thrust of Smart 1's ion drive is 0.07 N. Since one milli-Newton is 1/1000-th of a Newton, one Newton is 1,000 mN.

$$0.07\,\mathrm{N} = 1,000 \times 0.07 = 70\,\mathrm{mN}$$

This can be compared with the liftoff thrust of the Ariane V rocket which is 15,360 kN. However, the ion drive is designed to run for a much longer time than the Ariane launch.

Click on the **green** square to return

□

**Exercise 5(d)** If the power of each of the ten turbines is $400\,\mathrm{kW}$, the total power is $4,000\,\mathrm{kW}$.

To express this in mega-Watts, recall that $1\,\mathrm{kW} = 10^3\,\mathrm{W}$ and $1\,\mathrm{MW} = 10^6\,\mathrm{W}$. Therefore

$$\begin{aligned} 1\,\mathrm{kW} &= \frac{10^3}{10^6}\,\mathrm{MW} \\ &= 10^{-3}\,\mathrm{MW} \end{aligned}$$

From this we have

$$\begin{aligned} 4,000\,\mathrm{kW} &= 4,000 \times 10^{-3}\,\mathrm{MW} \\ &= 4\,\mathrm{MW} \end{aligned}$$

So the total power of this, the UK's first commercial wind farm, is $4\,\mathrm{MW}$.

Click on the **green** square to return □

# Solutions to Quizzes

**Solution to Quiz:** The Mars Climate Orbiter was launched by NASA in 1998 and crashed into the planet Mars in 1999. An investigation concluded that a mix up about the units of the force of correction thrusters was responsible for the spacecraft hitting the Martian atmosphere and breaking up.

NASA uses SI units but the Lockheed corporation which made the thrusters used English units. The conversion factor was forgotten and so the 'correction' rocket firings caused the expensive probe to be destroyed. End Quiz

**Solution to Quiz:** Density is mass per unit volume. The units of mass are kg and the units of volume are $m^3$. So again using the rules treated in the package on powers we have

$$\begin{aligned} \text{Units of density} &= \frac{\text{Units of mass}}{\text{Units of volume}} \\ &= \frac{\text{kg}}{\text{m}^3} \\ &= \text{kg}\,\text{m}^{-3} \end{aligned}$$

The SI units of density are kilograms per metre cubed. End Quiz

**Solution to Quiz:** Frequency is the number of oscillations per second. The number of oscillations is just a number and does not depend on any choice of units. The units of time are seconds:

$$\begin{aligned}\text{Units of frequency} &= \frac{1}{\text{Units of time}}\\ &= \frac{1}{\mathrm{s}}\\ &= \mathrm{s}^{-1}\end{aligned}$$

The SI unit of frequency is cycles per second. This unit is also called a **Hertz** (Hz). End Quiz

**Solution to Quiz:** Acceleration is the rate of change of speed with time. The units of speed are $\mathrm{m\,s^{-1}}$ and the units of time are $s$:

$$\begin{aligned} \text{Units of acceleration} &= \frac{\text{Units of speed}}{\text{Units of time}} \\ &= \frac{\mathrm{m\,s^{-1}}}{\mathrm{s}} \\ &= \mathrm{m\,s^{-2}} \end{aligned}$$

The SI unit of acceleration is metres per second per second. This can also be said as *metres per second squared*.

A constant acceleration of $10\,\mathrm{m\,s^{-2}}$ means that the speed of the object increases by $10\,\mathrm{m\,s^{-1}}$ every second. End Quiz

**Solution to Quiz:** From the equation for kinetic energy, $E = \frac{1}{2}mv^2$, we see that its units must be

$$\text{Units of energy} = \text{Units of mass} \times \text{units of speed}^2$$

The unit of speed is $\mathrm{m\,s^{-1}}$, so:

$$\begin{aligned} \text{Units of energy} &= \mathrm{kg} \times (\mathrm{m\,s^{-1}})^2 \\ &= \mathrm{kg\,m^2 s^{-2}} \end{aligned}$$

One Joule is therefore one kilogram metre squared per second squared.

End Quiz

**Solution to Quiz:** The frequency of the X-ray is given by the speed of light ($3 \times 10^8 \mathrm{m\,s^{-1}}$) over the wavelength ($6 \times 10^{-10}\,\mathrm{m}$):

$$\begin{aligned}\nu &= \frac{c}{\lambda} = \frac{3 \times 10^8\,\mathrm{m\,s^{-1}}}{6 \times 10^{-10}\,\mathrm{m}} \\ &= 0.5 \times 10^{8+10}\,\mathrm{s^{-1}} = 5 \times 10^{17}\,\mathrm{s^{-1}}\end{aligned}$$

This can also be written as $5 \times 10^{17}$ Hz. It means that an X-ray oscillates $5 \times 10^{17}$ times every second! End Quiz

**Solution to Quiz:** Pressure is force per unit area. So if a force of 50 N is applied to an area of $0.002\,\mathrm{m}^2$:

$$\begin{aligned}\text{Pressure} &= \frac{50\,(\mathrm{N})}{0.002\,(\mathrm{m}^2)}\\ &= \frac{50}{2\times 10^{-3}}\,\mathrm{N\,m^{-2}}\\ &= \frac{50\times 10^{3}}{2} = 25,000\,\mathrm{Pa}\end{aligned}$$

This is 25 kPa. End Quiz