Maths for Science

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PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

The Open University

Contents

Introduction

Welcome to *Maths for Science*. There are many reasons for studying maths and a compelling motivation for many people is that it provides a way of representing and investigating the nature of the real world. Real world contexts could include population statistics, or economics, or engineering. Here, the context is 'science' in its broadest sense.

Much of science is couched in the language of mathematics. Nearly all courses in science will assume some mathematical skills and techniques. It is clearly not possible for *Maths for Science* to discuss all the mathematical techniques you might need to pursue your study of science to degree level, but by the end of it you will have acquired a good array of basic mathematical tools and confidence in using them. Equally importantly, you will have a foundation that should make it much easier to learn further mathematics if and when required.

Maths is in some sense a language with its own alphabet, vocabulary and 'rules of grammar'. With any language the only route to fluency is use and practice, but eventually the process of constructing or understanding sentences becomes automatic and one can then concentrate wholly on the message behind the words. You should aim to develop a similar confidence and fluency in carrying out certain important mathematical operations. There are few shortcuts: the route requires practice, practice and more practice! Keep paper, a pencil and your calculator to hand as you study, and use them constantly. You may find it helpful to write out notes and even to rework some of the examples given in the text as you go along. You will see that there are lots of questions seeded through the text and at the ends of sections; *you should work through each question as you reach it*. Links are provided to the solutions, but don't be tempted to look at these until you have made a serious attempt at working out the answer for yourself. If you have solved all parts of a question successfully on your own, then you are ready to move on.

The focus of *Maths for Science* is maths and not science, so you are not expected to bring specific prior knowledge of any particular branch of science. However, most of the examples and questions involve the application of mathematical tools to a real scientific purpose, so you will probably discover some interesting science along the way. Enjoy the journey!

1 Starting Points

The point to start from is always what you already know. It is assumed that you are familiar with the everyday usage of the basic arithmetic operations of addition, subtraction, multiplication, division and the use of a calculator to carry them out, decimal notation (e.g. for money), the representation of an idea by a formula (such as Einstein's famous $E = mc^2$), and the interpretation of information on a chart or graph (of the kind that might, for instance, accompany a TV news item about economic trends). Beyond that, you will find that many of the early chapters begin with a little revision of ideas and skills that you will probably already have met. This chapter, which concentrates on ideas about numbers – including fractions and powers – and the use of your calculator, is slightly different from later ones in that it covers concepts that are the basis for what is to follow in the rest of the course, so more of it may constitute revision.

If the points covered in the rest of this chapter are completely familiar, you need not spend very long on them, but they are worth checking out thoroughly as they are the foundation of much that is to come later in *Maths for Science*. Even if it is only for the sake of revision, make sure you understand all the emboldened terms and test your own skills against the learning outcomes by doing the numbered questions. If any of the material is new to you, time spent mastering it now will pay rich dividends later.

1.1 Numbers

'Numbers rule the universe' (Pythagoras)

Numbers are the bedrock of mathematics, underlying measurement, calculation and statistics, among other branches of maths. Everybody is familiar with the counting numbers (1, 2, 3, etc.), but scientists also make use of other kinds of numbers, so it is appropriate to begin this course with some revision of numbers of various sorts and the ways in which they may be combined.

1.1.1 Different types of number

One convenient way to represent numbers is on a 'number line', as shown in Figure 1.1. A 'step' to the right is taken by adding 1 to the previous number and a step to the left by subtracting 1. Positive and negative whole numbers, including zero, are called integers.

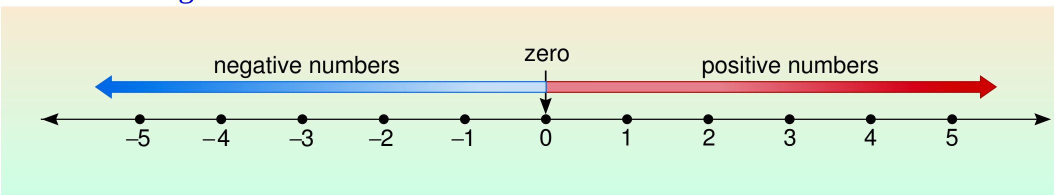


Figure 1.1: A number line showing the integers from −5 to 5.

Fractions (formed by dividing one integer by another) and their equivalent decimal numbers fit on the number line between the integers. For example, (i.e. 0.5) is halfway between 0 and 1, and −2.5 is halfway between −2 and −3. A number in which there is a decimal point (e.g. 0.5, 2.5, 100.35, etc.) is said to be written in decimal notation.

Figure 1.2 shows part of a thermometer. The inset portion covers a range from about +4.4 °C to −5.6 °C, which might represent the variation in temperature over a 24-hour period during the winter in the UK.

This illustrates how subdivision of the number line forms the basis of a scale for measuring physical quantities that can vary continuously. In this case, the scale between the integral values is divided into tenths. (Note that, in order to describe a physical quantity the numerical value has to be accompanied by a unit of measurement, in this case the degree Celsius. This aspect of measuring is covered in Chapters 2 and 3.)

In the case of a fraction such as $\frac{213}{25}$, the decimal equivalent is exact to two places of decimals (i.e. two digits after the decimal point):

$$\frac{213}{25} = 8.52$$

This decimal equivalent of $\frac{213}{25}$ cannot be given to more than two places of decimals except by putting zeros on the end (e.g. 8.520 000), so it is said to terminate at the digit 2.

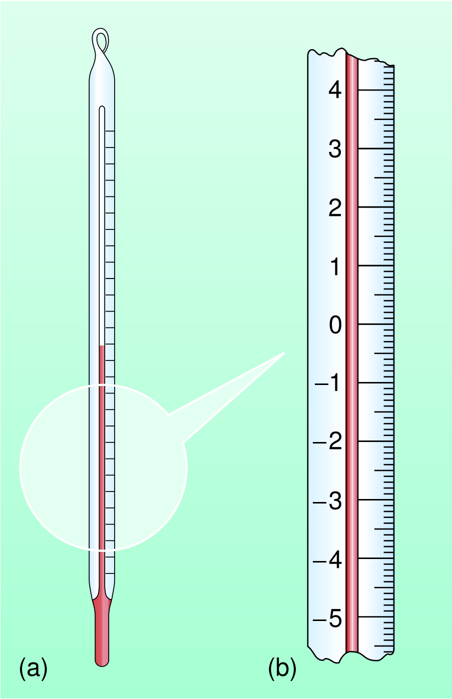


Figure 1.2: Part of a thermometer.

However, if you work out a fraction like $\frac{1}{3}$ on your calculator you will get a decimal like 0.333 333 333 (the number of digits displayed will depend on the make of your calculator). $\frac{41}{333}$ will come out as 0.123 123 123, and $\frac{70}{9}$ as 7.777 777 778. These decimals in fact recur (i.e. repeat themselves) for ever, so they are called infinite recurring decimals. The calculator truncates them when it runs out of digits on the display, and in the case of the final example also ‘rounds up’ the last digit from a 7 to an 8. In scientific calculations, it is usually totally inappropriate to quote so many digits after the decimal point and in Chapter 2 we will consider the rules for deciding how to round off such numbers in real situations.

Fractions and decimals are grouped together as the so-called rational numbers. All the rational numbers result in a decimal that either terminates or recurs. However, there are also numbers whose decimal equivalent neither terminates nor recurs. These numbers cannot be obtained by dividing one integer by another, so they are called irrational numbers. Well-known examples are $\sqrt{2}$ (the number that multiplied by itself gives 2, said as ‘the square root of 2’) and π, which is defined as the number obtained by dividing the circumference of a circle by its diameter. This would be an appropriate moment to check that you know how to use the π button on your calculator. You should be able to get:

$$2 \times \pi = 6.283\,185\,307$$

Note that as there are so many makes of scientific and graphics calculators on the market, each operating differently, it is impossible to state the exact sequence of keystrokes you will need to carry out particular calculations. Whenever you meet a new type of mathematical operation, you should therefore check that you know how to perform it on your own calculator and refer to the manufacturer's instruction book if necessary. A calculator symbol in the margin will alert you to the points at which you particularly need to carry out this kind of check.

All the integers, rational and irrational numbers can be placed somewhere on the number line, so they are grouped together as the real numbers. All the numbers you will use in this course will be real. However, it may interest you to know that there are also imaginary numbers based on the square root of minus 1, which is usually represented by the symbol i. Numbers made up of real and imaginary parts, such as $(3 + 2i)$ are known as complex numbers. Complex numbers are used quite extensively in science and have practical applications in telecommunications, electrical engineering and the beautiful patterns of fractals.

In case hearing about all these different types of numbers leads you to think that straightforward 'counting numbers' hold little interest for scientists, Box 1.1 shows how a series of numbers, which mathematicians find interesting in their own right, have also been found to describe intricate patterns of plant growth.

Box 1.1 Fibonacci numbers

The sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \ldots$$

was first defined in 1202 by the Italian mathematician Leonardo of Pisa, nicknamed Fibonacci. Each term in the sequence after the first two is obtained by adding together the previous two ($0+1 = 1$; $1+1 = 2$; $1+2 = 3$; $2 + 3 = 5$, etc.)

It is intriguing to discover that the spiral patterns of plant growth correspond to pairs of numbers in this series, as illustrated in Figure 1.3.

Part (a) shows a pinecone with 8 parallel rows of bracts spiralling gradually and 13 parallel rows of bracts spiralling steeply.

Part (b) shows a sunflower head in which the seeds spiral out from the centre: 55 rows clockwise and 89 rows anticlockwise.

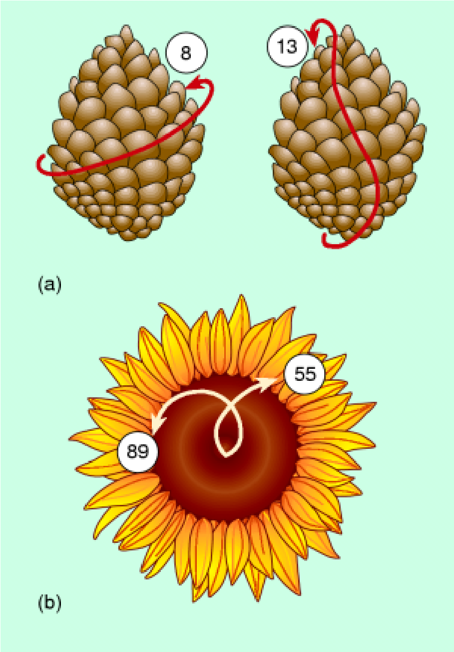


Figure 1.3: Fibonacci numbers in nature.

1.1.2 Calculating with negative numbers

Many scientific quantities can take negative values. For example, chemical reactions may either give out heat to the surroundings or absorb heat from the surroundings. Scientists adopt a convention that in the case of a heat-absorbing reaction, the change in energy has a positive value. In the case of a heat-releasing reaction (such as combustion), on the other hand, the energy change is negative. To be able to handle quantities like this in scientific calculations it is essential to understand the rules for performing the arithmetic operations (addition, subtraction, multiplication and division) when negative numbers are involved. If I amalgamate a credit card debt of £100 with an overdraft of £150, I owe £250 in total:

$$\text{£}100\,\text{debt} + \text{£}150\,\text{debt} = \text{£}250\,\text{debt}$$

Just in terms of numbers, this is equivalent to writing:

$$(-100) + (-150) = -250$$

Note from this example how brackets can be used to make it clear how numbers and signs are associated. The rules for performing arithmetic operations with negative numbers are summarized by the examples in the box 'Arithmetic with negative numbers'. You should check that you are familiar with all the rules exemplified in the box.

Arithmetic with negative numbers

The numbers used as examples here are small integers between 1 and 10, but could of course be any number. As is normally the case, positive numbers are not preceded by a + sign.

$$
\begin{array}{lll}
(-3) + 5 = 2 & 3 + (-4) = -1 & (-3) + (-3) = -6 \\
(-5) - 2 = -7 & 4 - (-3) = 7 & (-5) - (-4) = -1 \\
(-2) \times 2 = -4 & 3 \times (-2) = -6 & (-2) \times (-2) = 4 \\
(-3) \div 3 = -1 & 3 \div (-3) = -1 & (-3) \div (-3) = 1
\end{array}
$$

Thinking about some of the examples in concrete terms may help to make sense of them. For instance, taking money from a bank account that is already overdrawn increases the amount of the debt (i.e. makes it 'more negative'). Doubling an overdraft produces an even larger debt (i.e. a bigger negative number).

Brackets are included to associate negative signs with particular numbers. For example, $3+(-4)$ means that (-4) is being added to 3; this is equivalent to subtracting 4 from 3, with the result (-1).

Before reading on, test your understanding of the rules by doing Question 1.1.

Question 1.1

Without using your calculator, work out:

(a) $(-3) \times 4$ Answer

(b) $(-10) - (-5)$ Answer

(c) $6 \div (-2)$ Answer

(d) $(-12) \div (-6)$ Answer

The examples given so far illustrate one important feature of both addition and multiplication: both these operations are commutative. This is just the mathematical way of saying that if one adds two numbers then the result (called the sum) is identical whichever number is written first. For example:

$$5 + 3 = 8 \text{ and } 3 + 5 = 8$$
$$(-2) + 3 = 1 \text{ and } 3 + (-2) = 1$$

Similarly, in multiplying two numbers the result (called the product) is unchanged if the order of the numbers is reversed. For instance:

$$5 \times 4 = 20 \text{ and } 4 \times 5 = 20$$
$$(-3) \times 4 = -12 \text{ and } 4 \times (-3) = -12$$

Subtraction and division, on the other hand, are not commutative:

$$5 - 3 = 2 \text{ but } 3 - 5 = -2$$

$$8 \div 4 = 2 \text{ but } 4 \div 8 = \tfrac{1}{2}$$

The commutativity of addition and multiplication may seem rather obvious when applied to the counting numbers, but is worthy of attention because of its importance in the algebraic manipulations that will be discussed in Chapter 4.

Worked example 1.1 and Question 1.2 are two rather more realistic examples requiring the use of arithmetic with negative numbers.

Worked example 1.1

One of the hottest places on Earth is Death Valley, California, where an air temperature of 56 °C has been recorded. Probably the coldest inhabited place is the Siberian village of Oymyakon, where the temperature has fallen to −72 °C. What is the difference in temperature between these two extremes?

Answer

The difference in temperature may be worked out in two ways. The first method involves subtracting the lower temperature from the higher, i.e. 56 °C − (−72 °C), which gives a *positive* difference of 128 Celsius degrees. This is the amount by which Death Valley is hotter than Oymyakon. Alternatively, it is equally valid to subtract the higher temperature from the lower, i.e. −72 °C − 56 °C, which gives a *negative* difference of −128 Celsius degrees. This is equivalent to saying that Oymyakon is 128 Celsius degrees colder than Death Valley.

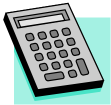
This example shows that in scientific calculations involving negative numbers it is important to keep the physical situation in mind.

Question 1.2 Answer

The maximum temperature range within the oceans is 31.9 Celsius degrees. This is a much smaller variation in temperature than that achievable for the air above a landmass, in part because the lowest ocean temperature is fixed at the temperature at which seawater freezes. The highest recorded ocean temperature is 30.0 °C. What is the freezing point of seawater?

1.1.3 Working with negative numbers on a calculator

The calculations in Questions 1.1 and 1.2 were easy enough to work out by hand, but many of the calculations you will encounter in science will require the use of a calculator. It is therefore important to check that you know how to input negative numbers into your own calculator.



Take the following examples:

$$6 + (-8) = -2$$
$$4 - (-3) = 7$$
$$5 \times (-3) = -15$$
$$(-8) \div (-2) = 4$$

and make sure that you can carry out each sum on your calculator, obtaining the correct sign on the display of the answer. With some makes of calculator you will be able to enter the expression on the left-hand side more or less as it is written, with or without brackets. With other makes you may have to use a combination of the arithmetic operation keys and the +/− (or on some makes ±) button.

When you are confident that you can input negative numbers in association with the first arithmetic operations, test your skill with Question 1.3.

Question 1.3

Making sure you input all the signs, use your calculator to work out the following:

(a) $117 - (-38) + (-286)$ Answer

(b) $(-1624) \div (-29)$ Answer

(c) $(-123) \times (-24)$ Answer

There is, however, one case in which the calculator does not fully deal with signs, and that case concerns square roots. The 'square root of 9' is defined as the number that multiplied by itself gives 9. One such number is 3:

$$3 \times 3 = 9$$

and if you use your calculator to work out $\sqrt{9}$ you will indeed obtain the answer 3. However, it is also true that

$$-3 \times -3 = 9$$

So the square root of 9 is either +3 or −3. It is a mathematical convention that the notation $\sqrt{9}$ means 'the positive value of the square root of 9', and this is what your calculator displays. In cases in which the negative value of the square root might be relevant this is indicated by use of the sign ± (plus or minus) before the square root sign, i.e. $\pm\sqrt{9}$.

In Section 1.1.1, the number $\sqrt{2}$ was given as an example of an irrational number. Check that you can use the square root button on your own calculator to get

$$\sqrt{2} = 1.414\,213\,562$$

(You may obtain more or fewer digits depending on the make and model of your calculator. The fact that the number is irrational means that in any case it never ends.)

Question

What is $\dfrac{\sqrt{5}}{3}$?

Answer

$$\frac{\sqrt{5}}{3} = 0.745\,355\,922$$

Be sure to check that you can obtain this value on your own calculator, by ensuring that the calculator takes the square root of 5 *before* dividing by 3. Otherwise, you will get the positive value of the square root of $\frac{5}{3}$, which is not the same at all!

$$\sqrt{\frac{5}{3}} = 1.290\,994\,449$$

1.1.4 The number zero

Zero is a number to be careful about, especially when it is used in multiplication or division.

If you try multiplying 0 by 6 on your calculator, you will get the answer 0. This is hardly surprising. If we start off with nothing, it doesn't matter how often we multiply it, we still have nothing. The commutativity of multiplication shows that 6×0 is therefore also equal to 0, and your calculator will confirm this.

The result of multiplying any number by 0 is 0.

In a similar way, dividing 0 by any non-zero number gives zero.

Trying to divide by zero is more problematic. If you enter $6 \div 0$ into your calculator, you will get an error message. To understand why, imagine dividing 6 by successively smaller and smaller numbers: the answers will get successively larger and larger. The number by which we're dividing approaches zero, the result of the division becomes too large for the calculator to cope with. Dividing by zero does not produce a meaningful number and is to be avoided!

1.2 Fractions

With the increasing decimalization of everyday units of measurement, we use fractions less than people used to. Nowadays adding eighths and sixteenths of inches is about as much as you might need to do, and that only if you still have a ruler, or some items in a toolbox, marked in inches. However the ability to add, subtract, multiply and divide using numerical fractions is extremely important in *Maths for Science*, because it is the basis for the skill of manipulating *algebraic* fractions which will be discussed in Chapter 4.

1.2.1 Using fractions

Fractions are characterized by a numerator (the number on top) and a denominator (the number on the bottom). So in the fraction $\frac{3}{8}$, the numerator is 3 and the denominator is 8.

A pictorial representation, such as that in Figure 1.4, makes it obvious that it is possible to have fractions which have different numerators and denominators, but are nevertheless equal. The cake can be divided into two and the shaded half further sub-divided into two quarters or four eighths, but half the cake still remains shaded. So the fractions $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{4}{8}$ all represent the same amount of the original cake, and can therefore be described as equivalent fractions.

Figure 1.4 exemplifies the most fundamental rule associated with fractions:

> The value of a fraction is unchanged if its numerator and denominator are both multiplied by the same number, or both divided by the same number.

In the case of the half cake, numerator and denominator have been multiplied by 2 to get the equivalent two quarters and again to get the equivalent four eighths. In the following example of equivalent fractions, other multiplying and dividing numbers have been used:

$$\frac{6}{9} = \frac{2}{3} = \frac{8}{12} = \frac{10}{15}$$

$\frac{2}{3}$ is the simplest form in which this fraction may be expressed, i.e. the one in which the numerator and denominator have the smallest possible value.

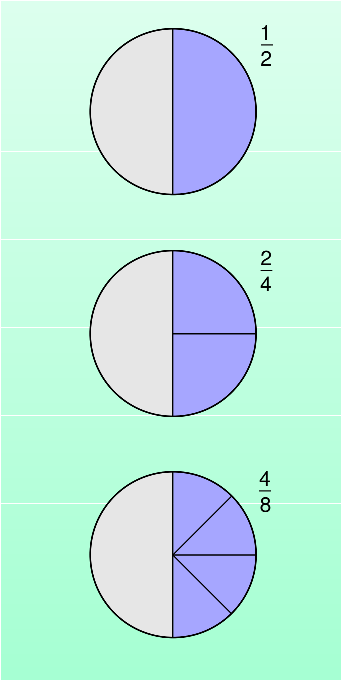


Figure 1.4: Sharing out half a cake.

A percentage means a ‘number of parts per hundred’, so is equivalent to a fraction in which the denominator is 100. For example, 50% is the same as $\frac{50}{100}$ or $\frac{1}{2}$

Question

Express 35% as a fraction of the simplest possible form.

Answer

35% is the same as $\frac{35}{100}$. The value of the faction will be unchanged if the numerator and denominator are both divided by the same number, and 35 and 100 can both be divided by 5. Doing this gives

$$\frac{35}{100} = \frac{7}{20}$$

This is the simplest form in which the fraction can be expressed.

One way to convert a fraction to a percentage is to multiply top and bottom of the fraction by whatever number is required to make the denominator equal to 100. For instance:

$$\frac{1}{4} = \frac{1 \times 25}{4 \times 25} = \frac{25}{100}$$

Hence $\frac{1}{4}$ is equivalent to 25%.

In the first few sections of this course, all fractions have been written in the form $\frac{3}{4}$. However, in most maths and science texts, you will find that the alternative form, 3/4, is also very common, so you have to become equally comfortable with both systems and also have to be able to swap between them at will. From now on, therefore, both notations will be used.

1.2.2 Adding and subtracting fractions

Suppose we want to add the two fractions shown below:

$$\frac{3}{4} + \frac{7}{16}$$

We cannot just add the 3 and the 7. The 3 represents 3 'quarters' and the 7 represents 7 'sixteenths', so adding the 3 to the 7 would be like trying to add 3 apples and 7 penguins!

In order to add or subtract two fractions, it is necessary for them both to have the same *denominator* (bottom line).

Fractions with the same denominator are said to have a common denominator. In numerical work, it is usually convenient to pick the smallest possible number for this denominator (the so-called lowest common denominator). In this example, the lowest common denominator is 16; we can multiply both top and bottom of the fraction $\frac{3}{4}$ by 4 to obtain the equivalent fraction $\frac{12}{16}$, so the calculation becomes

$$\frac{3}{4} + \frac{7}{16} = \frac{12}{16} + \frac{7}{16} = \frac{19}{16}$$

A top heavy fraction such $\frac{19}{16}$ (i.e. one in which the numerator is larger than the denominator) is sometimes referred to as an improper fraction. We could also write the final answer as $1\frac{3}{16}$. This notation is called a mixed number (i.e. a combination of a whole number and a simple fraction). However for most purposes in this course it is better to leave things as improper fractions.

If the lowest common denominator is not easy to spot, it is perfectly acceptable to use *any* common denominator when adding and subtracting fractions. It may be most convenient to multiply the top and bottom of the first fraction by the denominator of the second fraction, and the top and bottom of the second fraction by the denominator of the first. A return to our example may make this clearer:

$$\frac{3}{4}+\frac{7}{16}=\frac{3\times 16}{4\times 16}+\frac{7\times 4}{16\times 4}=\frac{48}{64}+\frac{28}{64}=\frac{76}{64}$$

However, $\frac{76}{64}$ is not the simplest form in which this fraction can be expressed. We can divide both the numerator and the denominator by four to obtain $\frac{19}{16}$. Reassuringly, this is the same answer as we obtained before!

This process of dividing the top and bottom of a fraction by the same quantity is often referred to as cancellation, because it is commonly shown by striking through the numbers being divided. For example, $\frac{5}{15}$ can be simplified by dividing the numerator and denominator by 3, and this may be shown as

$$\frac{\cancel{5}^{\,1}}{\cancel{15}_{\,3}}$$

Worked example 1.2

Evaluate $\frac{3}{2}+\frac{1}{32}$, giving the answer in the form of the simplest possible improper fraction.

Note that the instruction to 'evaluate' simply means 'calculate the value of'.

Answer

Choosing 2×32 as the common denominator,

$$\begin{aligned}\frac{3}{2}+\frac{1}{32} &= \frac{3 \times 32}{2 \times 32}+\frac{1 \times 2}{32 \times 2}\\ &= \frac{96}{64}+\frac{2}{64}\\ &= \frac{98}{64}\\ &= \frac{\cancel{98}^{49}}{\cancel{64}_{32}}\end{aligned}$$

This cannot be simplified any further, so

$$\frac{3}{2}+\frac{1}{32}=\frac{49}{32}$$

Question 1.4

Without using a calculator, evaluate the following, leaving your answers in the form of the simplest possible fractions.

(a) $\frac{2}{3} - \frac{1}{6}$ Answer

(b) $\frac{1}{3} + \frac{1}{2} - \frac{2}{5}$ Answer

(c) $\frac{5}{28} - \frac{1}{3}$ Answer

1.2.3 Manipulating fractions

It is very important to remember that multiplying both numerator and denominator by the same non-zero number, or dividing both numerator and denominator by the same non-zero number, are the *only* things you can do to a fraction that leave its value unchanged. Adding the same number to the numerator and denominator will alter the value of the fraction, as will any other operations. The following question will help you to convince yourself of this, so it is particularly important that you should work through it at this point.

Question 1.5

Take any fraction, say $\frac{4}{16}$, and evaluate it as a decimal, using your calculator if necessary. Now try each of the following operations in turn, using your calculator to work out the result:

(a) choose any integer and add it to the numerator and denominator Answer

(b) subtract the same integer from the numerator and denominator Answer

(c) square the numerator and the denominator (i.e. multiply the numerator by itself, and the denominator by itself) Answer

(d) take the square root of the numerator and the square root of the denominator. Answer

The results you obtained for Question 1.5 confirm that, for example, adding the same non-zero number to the top and bottom of a fraction changes its value, as do operations such as taking the square root of the numerator and denominator. The experience of all calculations of this type can be generalized by saying that *excluding operations involving the integer zero,*

A fraction is unchanged by either the multiplication, or the division, of its numerator and denominator by the same amount. All other operations carried out on the fraction will alter its value.

In terms of numerical fractions, this rule may seem fairly obvious. But forgetting it once the numbers are replaced by symbols is the root cause of many errors in algebra!

1.2.4 Multiplying fractions

The expression ‘three times two’ just means there are three lots of two (i.e. 2+2+2). So multiplying by a whole number is just a form of repeated addition. For example,

$$3 \times 2 = 2 + 2 + 2$$

This is equally true if you are multiplying a fraction by a whole number:

$$3 \times \frac{4}{5} = \frac{4}{5} + \frac{4}{5} + \frac{4}{5} = \frac{12}{5}$$

We could write the 3 in the form of its equivalent fraction $\frac{3}{1}$ and it is then clear that the same answer is obtained by multiplying the two numerators together and the two denominators together.

$$\frac{3}{1} \times \frac{4}{5} = \frac{3 \times 4}{1 \times 5} = \frac{12}{5}$$

In fact, this procedure holds good for any two fractions.

To multiply two or more fractions, multiply the numerators (top lines) together and also multiply the denominators (bottom lines) together.

So

$$\frac{3}{4} \times \frac{7}{8} = \frac{3 \times 7}{4 \times 8} = \frac{21}{32}$$

Multiplying three fractions together is done by simple extension of the method used in the previous examples:

$$\frac{7}{16} \times \frac{7}{8} \times \frac{3}{4} = \frac{7 \times 7 \times 3}{16 \times 8 \times 4} = \frac{147}{512}$$

1.2.5 Dividing fractions

How are we to interpret $4 \div \frac{1}{2}$? The analogy with dividing by an integer may help. The expression $4 \div 2$ asks us to work out how may twos there are in 4 (answer 2). In exactly the same way, the expression $4 \div \frac{1}{2}$ asks how many halves there are in 4. Figure 1.5 illustrates this in terms of circles. Each circle contains two half-circles, and 4 circles therefore contain 8 half-circles. So

$$4 \div \frac{1}{2} = 4 \times 2 = 8$$

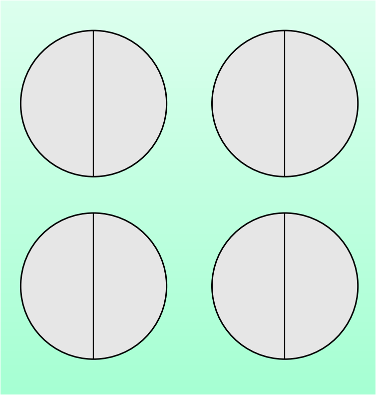


Figure 1.5: Each circle contains two half-circles.

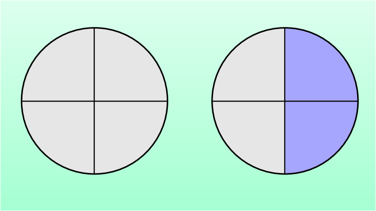


Figure 1.6: Each half-circle contains two quarter-circles.

Similarly, $\frac{1}{2} \div \frac{1}{4}$ asks how many quarters there are in a half. Figure 1.6 illustrates that:

- each whole circle contains 4 quarter-circles
- each half-circle contains $\frac{1}{2} \times 4$ quarter-circles

So

$$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times 4 = \frac{1}{2} \times \frac{4}{1} = \frac{1 \times 4}{2 \times 1} = \frac{4}{2} = 2$$

This may be extended into a general rule

To divide by a fraction, turn it upside down and multiply.

So

$$
\begin{aligned}
\frac{4}{3} \div \frac{5}{9} &= \frac{4}{3} \times \frac{9}{5} \\
&= \frac{\cancel{36}^{12}}{\cancel{15}_{5}} \\
&= \frac{12}{5}
\end{aligned}
$$

Here the cancellation has been done by dividing the numerator and the denominator of the final answer by 3. However, cancellation could equally well have been carried out at an earlier stage,

$$
\frac{4}{\cancel{3}_{1}} \times \frac{\cancel{9}^{3}}{5} = \frac{12}{5}
$$

Note that divisions involving fractions are commonly written in several different ways; the example above might equally well have been expressed as $\frac{4}{3}\Big/\frac{5}{9}$ or $\frac{4/3}{5/9}$.

It is always important to remember that an integer is equivalent to a fraction in which the numerator is equal to that integer and the denominator is equal to 1: for example, the integer 3 is equivalent to the fraction $\frac{3}{1}$. So dividing by the integer 3 is equivalent to dividing by the fraction $\frac{3}{1}$, and that, according to the general rule about how to divide by a fraction, is the same as multiplying by the fraction $\frac{1}{3}$.

Thus $\dfrac{1}{2} \div 3 = \dfrac{1}{2} \div \dfrac{3}{1} = \dfrac{1}{2} \times \dfrac{1}{3} = \dfrac{1 \times 1}{2 \times 3} = \dfrac{1}{6}$

In this context, it may be helpful to restate the general rule in terms of a specific example:

Multiplying by $\frac{1}{2}$ is equivalent to dividing by 2.

Dividing by $\frac{1}{2}$ is equivalent to multiplying by 2.

The blue box and the cartoon use the integer 2 as the example, but it could of course be replaced by any other integer: it is equally true to say that dividing by $\frac{1}{10}$ is equivalent to multiplying by 10.

Question 1.6

Work out each of the following, leaving your answer as the simplest possible fraction:

(a) $\frac{2}{7} \times 3$ Answer

(b) $\frac{5}{9} \div 7$ Answer

(c) $\frac{1/6}{1/3}$ Answer

(d) $\frac{3}{4} \times \frac{7}{8} \times \frac{2}{7}$ Answer

1.3 Powers, reciprocals and roots

1.3.1 Powers

Most people are familiar with the fact that 2×2 can also be written as 2^2 (said as ‘two squared’) and $2 \times 2 \times 2$ as 2^3 (said as ‘two cubed’). This shorthand notation can be extended indefinitely, so $2 \times 2 \times 2 \times 2 \times 2 \times 2$ becomes 2^6 (said as ‘two raised to the power of six’ or ‘two to the power of six’, or more usually just as ‘two to the six'). In these examples, 2 is called the base number and the superscript indicates the number of '2's that have been multiplied together. The superscript number is variously called the exponent, the index (plural indices) or the power. In the rest of this section, the term exponent will be the one used, because that ties in most closely with the notation on calculators.

'Power' is a slightly confusing term because it is commonly used to denote two different quantities:

- the value of the superscript number (as in 'two to the power of six'),
- the complete package of base number and exponent .

The context should make it clear what is meant in any particular example.

In the following example, the base number is 5:

Exponent	1	2	3	4
Power of 5	5^1	5^2	5^3	5^4
Value	5	25	125	625

If you read this table starting at the right and stepping to the left, each time you take a step you are subtracting 1 from the number in the top row and dividing the number in the bottom row by five. On the basis of this pattern, mathematicians extend this table further to the left by continuing to apply the same 'rule' for each step, giving:

Exponent	-3	-2	-1	0	1	2	3	4
Power of 5	5^{-3}	5^{-2}	5^{-1}	5^{0}	5^{1}	5^{2}	5^{3}	5^{4}
Value	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25	125	625

Firstly, note the extremely important result that $5^0 = 1$.

Any base number raised to the power of zero is equal to 1.

Next, notice that $5^{-2} = \frac{1}{25}$. But since $25 = 5^2$, $\frac{1}{25}$ is also $\dfrac{1}{5^2}$. So we have developed a new form of shorthand such that

$$5^{-1} = \frac{1}{5} \qquad 5^{-2} = \frac{1}{5^2} \qquad 5^{-3} = \frac{1}{5^3} \qquad \text{and so on.}$$

Another way of saying this is that 5^{-2} is the reciprocal of 5^2. The reciprocal of any number is 1 divided by that number. Note that this also works the other way round: 5^2 is the reciprocal of 5^{-2}. This means that $\dfrac{1}{5^{-2}} = 5^2$.

The system shown above for powers of 5 could be applied to any base number, and is especially useful when applied to powers of ten, because then it ties in with our normal system for writing decimal numbers. In the example below, the table is constructed the other way round to emphasise this:

	thousands	hundreds	tens	units	point	tenths	hundredths	thousandths
Value	1000	100	10	1	.	0.1	0.01	0.001
Power of 10	10^3	10^2	10^1	10^0		10^{-1}	10^{-2}	10^{-3}
Exponent	3	2	1	0		−1	−2	−3

In the next chapter, you will see how useful this powers of ten notation can be in scientific work.

Question 1.7

Without using a calculator, evaluate

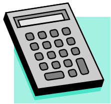
(a) 2^{-2} Answer

(b) $\dfrac{1}{3^{-3}}$ Answer

(c) $\dfrac{1}{4^0}$ Answer

(d) $\dfrac{1}{10^4}$ Answer

Your calculator probably has an x^2 button, and either an x^{-1} or a $1/x$ button, but to evaluate other powers you will have to use a special 'powers' button. On some calculators this is marked x^y, on others it has the symbol $\wedge$. To input a negative exponent, you may have to combine the powers button with the $+/-$ button. Make sure at this point that you can operate your own calculator to obtain correctly:



$5^4 = 625$

$5^{-1} = 0.2$ (i.e. 1/5)

$5^{-2} = 0.04$ (i.e. 1/25)

Question 1.8

Use your calculator to evaluate:

(a) 2^9 Answer

(b) 3^{-3} Answer

(c) $\dfrac{1}{4^2}$ Answer

Box 1.2 An intimate knowledge of powers!

Srinivasa Ramanujan (1887–1920), an Indian mathematician of immense talent, came to England in 1913 at the invitation of the distinguished British mathematician, G. H. Hardy. In his biography of Ramanujan, Hardy wrote:

I remember once going to see him when he was lying ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."

Indeed: $1729 = 1^3 + 12^3 = 9^3 + 10^3$

1.3.2 Multiplying and dividing with powers

In scientific calculations, it is very common to have to multiply and divide by powers, especially powers of ten. It is therefore extremely important to become confident in manipulating powers in this way, both with and without a calculator. However, the rules for doing so are quite easy to work out.

Suppose we wanted to multiply 10^3 by 10^2. We could write this out more fully as

$$10^3 \times 10^2 = (10 \times 10 \times 10) \times (10 \times 10) = 10^5$$

The exponent of the result (5) is the same as the sum of the two original exponents $(3 + 2)$.

The process is of course not limited to powers of ten. It works for any base number. For example:

$$2^2 \times 2^4 = (2 \times 2) \times (2 \times 2 \times 2 \times 2) = 2^6$$

Again, the exponent of the result (6) is the same as the sum of the two original exponents $(2 + 4)$.

The process also works for negative exponents. For example, since $5^{-2} = \dfrac{1}{5^2}$

$$5^3 \times 5^{-2} = (5 \times 5 \times 5) \times \frac{1}{5 \times 5} = 5 = 5^1$$

Adding the exponents here again gives the exponent of the answer:

$$3 + (-2) = 1$$

In science and maths, general rules are often stated in terms of symbols. We could express the rule we have discovered through the above examples in the much more general form

$$N^a \times N^b = N^{a+b} \tag{1.1}$$

where N represents any base number and a and b represent any exponents

Quantities such as those represented by the symbols N, a and b, which can take any value we choose, are called variables.

The example involving a negative exponent we looked at previously shows immediately how to extend the rules to cover situations in which we want to divide powers. We had:

$$5^3 \times 5^{-2} = 5^{3+(-2)} = 5^1 = 5$$

But as you will remember from Section 1.2.5, multiplying by a fraction is the same as dividing by that fraction turned upside down (i.e. its reciprocal). So multiplying by 5^{-2} is the same as dividing by its reciprocal (5^2), and we can write

$$5^3 \div 5^2 = 5^{3-2} = 5^1 = 5$$

This time, instead of adding the exponents, we have subtracted the second from the first. More generally,

$$N^a \div N^b = N^{a-b} \tag{1.2}$$

where N represents any base number and a and b represent any exponents

Question 1.9

Without using a calculator, simplify the following to the greatest possible extent (leaving your answer expressed as a power).

(a) $2^{30} \times 2^2$ Answer

(b) $3^{25} \times 3^{-9}$ Answer

(c) $10^2/10^3$ Answer

(d) $10^2/10^{-3}$ Answer

(e) $10^{-4} \div 10^2$ Answer

(f) $\dfrac{10^5 \times 10^{-2}}{10^3}$ Answer

1.3.3 Powers of powers

Consider now what happens when a number which is already raised to a power, for example 3^2, is again raised to a power. Suppose for example 3^2 is itself cubed, so that we have $\left(3^2\right)^3$. Writing this out in full shows that

$$\left(3^2\right)^3 = (3^2) \times (3^2) \times (3^2) = (3 \times 3) \times (3 \times 3) \times (3 \times 3) = 3^6$$

This time the exponents have been multiplied together to obtain the exponent of the answer: $3 \times 2 = 6$.

More generally,

$$(N^m)^n = N^{m \times n} \tag{1.3}$$

where N represents any base number and m and n represent any exponents

Equation 1.3 applies for all values of N, m and n whether positive or negative. So for example:

$$\left(\frac{1}{10^{20}}\right)^3 = \left(10^{-20}\right)^3 = 10^{(-20)\times 3} = 10^{-60} = \frac{1}{10^{60}}$$

This is equivalent to saying that

$$\left(\frac{1}{10^{20}}\right)^3 = \frac{1^3}{(10^{20})^3} = \frac{1}{10^{20\times 3}} = \frac{1}{10^{60}}$$

Question 1.10

Without using a calculator, simplify the following to the greatest possible extent, leaving your answer expressed as a power.

(a) $\left(4^{16}\right)^{2}$ Answer

(b) $\left(5^{-3}\right)^{2}$ Answer

(c) $\left(10^{25}\right)^{-1}$ Answer

(d) $\left(\dfrac{1}{3^{3}}\right)^{6}$ Answer

1.3.4 Roots and fractional exponents

Finally, how are we to interpret a power with a fractional exponent, such as $2^{1/2}$? The rule for multiplying powers gives a clue. Suppose we were to multiply $2^{1/2}$ by itself. Applying Equation 1.1 suggests that:

$$2^{1/2} \times 2^{1/2} = 2^{\left(\frac{1}{2}+\frac{1}{2}\right)} = 2^1 = 2$$

But the positive number that multiplied by itself gives 2 is more commonly written as $\sqrt{2}$. The two shorthands, $2^{1/2}$ and $\sqrt{2}$ are often used interchangeably.

Similarly, the number that multiplied by itself three times gives 125 is sometimes written as $\sqrt[3]{125}$ (said as 'the cube root of 125'), but more commonly written in science as $(125)^{1/3}$. This number is clearly 5, and you should notice the correspondence:

$$5^3 = 125 \text{ and conversely } (125)^{1/3} = 5$$

More generally,

The positive nth root of a number N can be written as either $\sqrt[n]{N}$ or as $N^{1/n}$

In practice, the first type of notation is only used when $n = 2$ or $n = 3$.

Worked example 1.3

Without using a calculator, evaluate $\dfrac{\left(2^{1/2}\right)^7}{(2^3)^{1/2}}$

Answer

From Equation 1.3

$$\left(2^{1/2}\right)^7 = 2^{\frac{1}{2}\times 7} = 2^{7/2} \quad \text{and} \quad \left(2^3\right)^{1/2} = 2^{3\times\frac{1}{2}} = 2^{3/2}$$

so

$$\frac{\left(2^{1/2}\right)^7}{(2^3)^{1/2}} = \frac{2^{7/2}}{2^{3/2}}$$

From Equation 1.2

$$\begin{aligned}\frac{2^{7/2}}{2^{3/2}} &= 2^{7/2} - 2^{3/2}\\ &= 2^{4/2}\\ &= 2^2\\ &= 4\end{aligned}$$

Equation 1.3 can now be used to bring meaning to a number like $27^{2/3}$.

Since $\frac{2}{3} = \frac{1}{3} \times 2$, applying Equation 1.3 shows that $27^{2/3} = (27^{1/3})^2$ i.e. the square of the cube root of 27. The cube root of 27 is 3, so $27^{2/3}$ is equal to 3^2 or 9.

Question 1.11

Without using a calculator, simplify the following to the greatest possible extent, expressing your answer as an integer or a decimal.

(a) $\left(2^4\right)^{1/2}$ Answer

(b) $\sqrt{10^4}$ Answer

(c) $100^{3/2}$ Answer

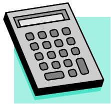
(d) $(125)^{-1/3}$ Answer

1.4 Doing calculations in the right order

In Section 1.1.2, brackets were used to make it clear that the minus signs were tied to particular numbers. Brackets can also be used to show the order in which calculations are to be performed.

If a calculation were written as

$$3 + 2 \times 5 =$$



should one do the addition first or the multiplication first? Try entering this expression into your calculator *exactly as it is written*. Do you get the answer 13? If so, your calculator knows the convention adopted by mathematicians everywhere that multiplication takes precedence over addition. The calculator has 'remembered' the 3 until it has worked out the result of multiplying 2 by 5 and has then added the 3 to the 10. According to the rules all mathematicians follow, if you wanted to add the 3 and the 2 first and then multiply that result by 5 you would have to write

$$(3 + 2) \times 5 = 25$$

Again, check that you can use the bracket function on your calculator to enter this expression exactly as written on the left-hand side of this equation and that you obtain the correct answer.

There are similar rules that govern the order of precedence of other arithmetic operations, which are neatly encapsulated in the mnemonic BEDMAS.

Order of arithmetic operations

Brackets take precedence over
Exponents. Then...
Division and
Multiplication must be done before...
Addition and
Subtraction.

So if we write $-3 - 12 \div 6$, the BEDMAS rules tell us we must do the division $(12 \div 6 = 2)$ before carrying out the subtraction $(-3 - 2 = -5)$. Try this on your calculator too; you may have to use the $+/-$ button to input the -3.

Many people, including scientists, find it hard to visualize the rules in a string of numbers. They often opt to use brackets to make things clear, even when those brackets simply reinforce the BEDMAS rules. So one could choose to write

$$(12 \div 3) + 2 = 6$$

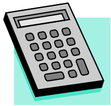
There is nothing wrong with adding such ‘redundant’ brackets — they are simply there for clarity and can even be entered into your calculator (try it). Far better to have a few additional brackets than to be confused about the order in which the calculation must be carried out!

There is one final quirk associated with the use of brackets. In mathematics, the multiplication sign is often left out (though its presence is implied) between numbers and brackets, and between brackets and brackets. So

$$2(3+1) = 2 \times (3+1) = 8$$

and

$$(1+1)(4+3) = 2 \times 7 = 14$$



Some calculators 'understand' this convention and some do not. Check your own calculator carefully using the two examples above.

The next operation in precedence after brackets involves exponents. If there are powers in the expression you are evaluating, deal with any brackets first, then work out the powers before carrying out any other arithmetical operations.

Question

Evaluate 2×3^2 and $(2 \times 3)^2$

Answer

In the first case, there are no brackets so the exponent takes precedence:

$$2 \times 3^2 = 2 \times 9 = 18$$

In the second case, the bracket takes precedence:

$$(2 \times 3)^2 = 6^2 = 36$$

Question 1.12

Evaluate (preferably without using your calculator):

(a) $35 - 5 \times 2$ Answer

(b) $(35 - 5) \times 2$ Answer

(c) $5(2 - 3)$ Answer

(d) 3×2^2 Answer

(e) $2^3 + 3$ Answer

(f) $(2 + 6)(1 + 2)$ Answer

1.5 Learning outcomes for Chapter 1

After completing your work on this chapter you should be able to:

1.1 carry out addition, subtraction, multiplication and division operations involving negative numbers;

1.2 add two or more fractions;

1.3 subtract one fraction from another;

1.4 multiply a fraction by an integer or by another fraction;

1.5 divide a fraction by a non-zero integer or by another fraction;

1.6 evaluate powers involving any base and positive, negative or fractional exponents;

1.7 multiply or divide two powers involving the same base;

1.8 evaluate any given power of a number already raised to a power.

2 Measurement in Science

Observation, measurement and the recording of data are central activities in science. Speculation and the development of new theories are crucial as well, but ultimately the predictions resulting from those theories have to be tested against what actually happens and this can only be done by making further measurements. Whether measurements are made using simple instruments such as rulers and thermometers, or involve sophisticated devices such as electron microscopes or lasers, there are decisions to be made about how the results are to be represented, what units of measurements will be used and the precision to which the measurements will be made. In this chapter we will consider these points in turn. Then in Chapter 3 we will go on to think about how measurements of different quantities may be combined, and what significance should be attached to the results.

2.1 Large quantities and small quantities

Scientists frequently deal with enormous quantities — and with tiny ones. For example it is estimated that the Earth came into being about four and a half thousand million years ago. It took another six hundred million years for the first living things — bacteria — to appear. Bacteria are so small that they bear roughly the same proportion to the size of a pinhead as the size that pinhead bears to the height of a four-year old child!

In the previous chapter, we saw how convenient powers of ten could be as a way of writing down very large or very small numbers. For example,

$$10^6 = 1\,000\,000 \text{ (a million) and } 10^{-3} = 1/1000 = 0.001 \text{ (a thousandth)}$$

This shorthand can be extended to any quantity, simply by multiplying the power of ten by a small number. For instance,

$$2 \times 10^6 = 2 \times 1\,000\,000 = 2\,000\,000 \text{ (two million)}$$

(The quantity on the left-hand side would be said as 'two times ten to the six'.)

Similarly,

$$3.5 \times 10^6 = 3\,500\,000 \text{ (three and a half million)}$$

$$7 \times 10^{-3} = 7/1000 = 0.007 \text{ (seven-thousandths)}$$

Scientists make so much use of this particular shorthand that it has come to be known as scientific notation (although in maths texts you may also find it referred to as standard index form or standard form.)

A quantity is said to be expressed in scientific notation if its value is written as a number multiplied by a power of ten. The number can be a single digit or a decimal number, but must be greater than or equal to 1 and less than 10.

Note the restriction: 75×10^2 is not in scientific notation and nor is 0.75×10^4, though these are both equivalent to 7.5×10^3 which *is* in scientific notation.

Scientific notation can be defined more succinctly by making use of some of the mathematical symbols denoting the relative sizes of quantities. These symbols are:

$>$ greater than (e.g. $3 > 2$);

$\geq$ greater than or equal to (e.g. $a \geq 4$ means that the quantity a may take the exact value 4 or any value larger than 4);

$<$ less than;

$\leq$ less than or equal to.

Note that '$a \geq 4$' and '$4 \leq a$' convey exactly the same information about the quantity a.

Using these symbols, scientific notation may be defined as a notation in which the value of a quantity is written in the form $a \times 10^n$, where n is an integer and $1 \leq a < 10$.

To move from scientific notation to integers or to decimal notation, first deal with the power of ten, then carry out the multiplication or division.

Worked example 2.1

Express the following numbers as integers or in decimal notation:

(a) 4.53×10^3

(b) 8.371×10^2

(c) 6.4×10^{-3}

Answer

(a) $4.53 \times 10^3 = 4.53 \times 1000 = 4530$

(b) $8.371 \times 10^2 = 8.371 \times 100 = 837.1$

(c) $6.4 \times 10^{-3} = 6.4 \times \dfrac{1}{1000} = \dfrac{6.4}{1000} = 0.0064$

Note that, as in Worked example 2.1, a requirement to express a quantity in a different form simply involves taking the quantity and writing down its equivalent in the new form. You may do this in one step, or write down intermediate steps as was done in the worked example.

Question 2.1

Without using your calculator, express the following numbers as integers or in decimal notation. Note that (a) and (b) are in scientific notation, while (c) is not.

(a) 5.4×10^4 Answer

(b) 2.1×10^{-2} Answer

(c) 0.6×10^{-1} Answer

Moving from an integer or decimal notation to scientific notation is equivalent to deciding what power of ten you need to multiply or divide by in order to convert the number you are starting with to a number that lies between 1 and 10.

Worked example 2.2

Express the following numbers in scientific notation:

(a) $356\,000$

(b) 49.7×10^4

(c) 0.831

Answer

(a) $356\,000 = 3.56 \times 100\,000 = 3.56 \times 10^5$

(b) $49.7 \times 10^4 = 4.97 \times 10 \times 10^4 = 4.97 \times 10^{(1+4)} = 4.97 \times 10^5$

(c) $0.831 = \dfrac{8.31}{10} = 8.31 \times 10^{-1}$

In this worked example, all the steps have been written out in full. You may be able to manage with fewer steps in your own calculations — just use as many or as few as you feel comfortable with in order to get the right answer!

Question 2.2

Without using your calculator, express the following numbers in scientific notation:

(a) 215 Answer

(b) 46.7 Answer

(c) 152×10^3 Answer

(d) 0.000 0876 Answer

It is only too easy to lose track of the sizes of things when using scientific notation, so you should make a habit of thinking carefully about what the numbers mean, bearing in mind that numbers may be positive or negative. For example:

-1×10^{10} is a very large negative number;

-1×10^{-10} is a very small negative number;

1×10^{-10} is a very small positive number.

Figure 2.1 places on the number line some numbers in scientific notation. You may find this helps you to visualize things.

We started this section thinking about the early Earth and the first appearance of life. Using scientific notation, the age of the Earth can be neatly expressed as 4.6×10^9 years and the size of one type of those early bacteria as 1.2×10^{-6} metres. Of course the value we come up with for such sizes will depend on the units in which we choose to make the measurements. If we were measuring the diameter of the Moon, we could elect to express it in metres or in kilometres, or even in miles.

2.2 Units of measurement

In the UK, two systems of units are in common use. We still use old imperial measures for some things: milk is sold in pints and signposts indicate distances in miles. But for many other everyday measurements metric units have been adopted: we buy petrol in litres and sugar in kilogram bags. A great advantage of metric units is that we no longer have to convert laboriously from imperial units, such as gallons, feet and inches, in order to trade with continental Europe. Also, calculations are easier in a metric (i.e. decimal) system! Similar advantages were the main consideration when in 1960 an international conference formally approved a standard set of scientific units, thus replacing at a stroke the many different systems of measurement that had been used up until then by scientists of different nationalities. This 'universal' system for scientific measurement is referred to as SI units (short for Système International d'Unités).

In SI, there are seven 'base units', which are listed in Box 2.1. Surprising as it may initially seem, every unit for every other kind of quantity (speed, acceleration, pressure, energy, voltage, heat, magnetic field, properties of radioactive materials, indeed whatever you care to name) can be made up from combinations of just these seven base units. For instance, speed is measured in metres per second. You will find some other combinations of base units described in Chapter 3. In this course we shall work mainly with the familiar base units of length, mass, time and temperature, and some of their combinations, but it is worth knowing that the other base units exist as you may meet them in other courses.

Box 2.1 The SI base units

Physical quantity	Name of unit	Symbol for unit
length	metre	m
time	second	s
mass	kilogram	kg
temperature	kelvin	K
amount of substance	mole	mol
electric current	ampere	A
luminous intensity	candela	cd

Most of these base units relate to physical descriptions that apply universally. The SI base unit of time, the second, is defined as the period over which the waves emitted by caesium atoms under specific conditions cycle exactly 9192 631 770 times. Then the SI base unit of length, the metre, is defined by stating that the speed of light in a vacuum, which is a constant throughout the Universe, is exactly 299 792 458 metres per second.

The SI base unit of mass, the kilogram, is the only fundamental unit that is defined in terms of a specific object. The metal cylinder which constitutes the world's 'standard kilogram' is kept in France. Note that the kilogram is actually the standard unit of *mass*, not of *weight*. In scientific language, the weight of an object is the downward pull on that object due to gravity, whereas its mass is determined by the amount of matter in it. When astronauts go to the Moon, where the pull of gravity is only about one-sixth of that on Earth, their mass remains the same but their weight drops dramatically! And in zero gravity, they experience a condition known as 'weightlessness'.

The SI base unit of temperature is the kelvin, which is related to the everyday unit of temperature, the degree Celsius:

$$(\text{temperature in kelvin}) = (\text{temperature in degrees Celsius}) + 273.15$$

(You will find some of the rationale for the kelvin scale of temperature in Chapter 5.)

The amount of a pure substance is expressed in the SI base unit of the mole. Whatever the smallest particle of a given substance is, one mole of that substance will contain $6.02211367 \times 10^{23}$ (known as Avogadro's number) of those particles. A mole of graphite contains Avogadro's number of carbon atoms. Carbon dioxide is made up of molecules in which one carbon atom is joined to two oxygen atoms, and a mole of carbon dioxide contains Avogadro's number of these molecules.

You will have noticed that while the base unit of length is the metre, not the kilometre, the base unit of mass is the kilogram, not the gram.

It is important to realize that, although in everyday usage it is common to say that you 'weigh so many kilos', there are two things wrong with this usage from the scientific point of view. First, as noted in Box 2.1, the kilogram is not a unit of weight, but a unit of mass. (The SI unit of weight, the newton, will be discussed in Chapter 3.) Secondly, in scientific language, 'kilo' is never used as an abbreviation for kilogram, in the sense of the everyday phrase 'he weighs so many kilos'. In science, kilo is always used as a *prefix*, denoting a thousand: one kilometre is a thousand metres, one kilogram is a thousand grams.

Another prefix with which everybody is familiar is 'milli', denoting a thousandth. One millimetre, as marked on ordinary rulers, is one-thousandth of a metre; or put the other way round, a thousand millimetres make up a metre. There are many other prefixes in use with SI units, all of which may be applied to any quantity. Like kilo and milli, the standard prefixes are based on multiples of 1000 (i.e. 10^3). The most commonly used prefixes are listed in Box 2.2.

It is important to write the symbols for units and their prefixes in the correct case. So k (lower case) is the symbol for the prefix ‘kilo’ whilst K (upper case) is the symbol for the Kelvin; m (lower case) is the symbol for the metre or the prefix ‘milli’ whilst M (upper case) is the symbol for the prefix ‘mega’.

Box 2.2 Prefixes used with SI units

prefix	symbol	multiplying factor
tera	T	10^{12} = 1000 000 000 000
giga	G	10^{9} = 1000 000 000
mega	M	10^{6} = 1000 000
kilo	k	10^{3} = 1000
–	–	10^{0} = 1
milli	m	10^{-3} = 0.001
micro	μ*	10^{-6} = 0.000 001
nano	n	10^{-9} = 0.000 000 001
pico	p	10^{-12} = 0.000 000 000 001
femto	f	10^{-15} = 0.000 000 000 000 001

* The Greek letter μ is pronounced ‘mew’.

The following data may help to illustrate the size implications of some of the prefixes:

- the distance between Pluto (the furthest planet in the Solar System) and the Sun is about 6 Tm,
- a century is about 3 Gs,
- eleven and a half days contain about 1 Ms,
- the length of a typical virus is about 10 nm,
- the mass of a typical bacterial cell is about 1 pg.

Astronomers have long been making measurements involving very large quantities, but scientists are increasingly probing very small quantities. 'Femtochemistry' is a rapidly developing area, which involves the use of advanced laser techniques to investigate the act of chemical transformation as molecules collide with one another, chemical bonds are broken and new ones are formed. In this work, measurements have to be made on the femtosecond timescale. Ahmed H. Zewail (whose laboratory at the California Institute of Technology in Pasadena is often referred to as 'femtoland') received the 1999 Nobel Prize in Chemistry for his development of this new area.

Although scientific notation, SI units and the prefixes in Box 2.2 are universal shorthand for all scientists, there are a few instances in which other conventions and units are adopted by particular groups of scientists for reasons of convenience. For example, we have seen that the age of the Earth is about 4.6×10^9 years. One way to write this would be 4.6 ‘giga years’ but geologists find millions of years a much more convenient standard measure. They even have a special symbol for a million years: Ma (where the ‘a’ stands for‘annum’, the Latin word for year). So in Earth science texts you will commonly find the age of the Earth written as 4600 Ma. It won’t have escaped your notice that the year is not the SI base unit of time — but then perhaps it would be a little odd to think about geological timescales in terms of seconds!

A few metric units from the pre-SI era also remain in use. In chemistry courses, you may come across the ångström (symbol Å), equal to 10^{-10} metres. This was commonly used for the measurement of distances between atoms in chemical structures, although these distances are now often expressed in either nanometres or picometres. Other metric but non-SI units with which we are all familiar are the litre (symbol l) and the degree Celsius (symbol °C).

There are also some prefixes in common use, which don’t appear in Box 2.2 because they don’t conform to the ‘multiples of 1000’ rule, but that when applied to particular units happen to produce a very convenient measure. One you will certainly have used yourself is centi (hundredth): rulers show centimetres (hundredths of a metre) as well as millimetres, and standard wine bottles are marked as holding 75 cl. One less commonly seen is deci (tenth) but that is routinely used by chemists in measuring concentrations of chemicals dissolved in water, or other solvents, as you will see in Chapter 3. In the next section you will also come across the decibel, which is used to measure the loudness of sounds.

Worked example 2.3

Diamond is a crystalline form of carbon in which the distance between adjacent carbon atoms is 0.154 nm. What is this interatomic distance expressed in picometres?

Answer

$$1\ \text{pm} = 10^{-12}\ \text{m so}$$

$$1\ \text{m} = \frac{1}{10^{-12}}\ \text{pm} = 10^{12}\ \text{pm}$$

$$1\ \text{nm} = 10^{-9}\ \text{m so}$$

$$\begin{aligned} 1\ \text{nm} &= 10^{-9} \times 10^{12}\ \text{pm} \\ &= 10^{-9+12}\ \text{pm} \\ &= 10^{3}\ \text{pm} \end{aligned}$$

$$\begin{aligned} 0.154\ \text{nm} &= 0.154 \times 10^{3}\ \text{pm} \\ &= 154\ \text{pm} \end{aligned}$$

Question 2.3

Using scientific notation, express:

(a) 3476 km (the radius of the Moon) in metres. Answer

(b) 8.0 μm (the diameter of a capillary carrying blood in the body) in nm, Answer

(c) 0.8 s (a typical time between human heartbeats) in ms. Answer

2.3 Scales of measurement

In thinking about the sizes of things, it is sometimes useful to do so in quite rough terms, just to the nearest power of ten. For example, 200 is nearer to 100 than it is to 1000, but 850 is nearer to 1000 than it is to 100. So if we were approximating to the nearest power of ten we could say 200 was roughly 10^2, but 850 was roughly 10^3. This process is called reducing the numbers to the nearest order of magnitude.

> The approximate value of a quantity expressed as the nearest power of ten to that value is called the order of magnitude of the quantity.

The easiest way to work out the order of magnitude of a quantity is to express it first in scientific notation in the form $a \times 10^n$. Then if a is less than 5, the order of magnitude is 10^n. But if a is equal to or greater than 5, the power of ten is rounded up by one, so the order of magnitude is 10^{n+1}. For example, the diameter of Mars is 6762 km. This can be written as 6.762×10^3 km, and because 6.762 is greater than 5, the diameter of Mars is said to be 'of order 10^4 km'.

This is normally written as:

$$\text{diameter of Mars} \sim 10^4 \text{ km}$$

where the symbol $\sim$ denotes 'is of order'.

Question

What is the order of magnitude of the mass of the Earth, 6.0×10^{24} kg?

Answer

Mass of the Earth $\sim 10^{25}$ kg (since 6.0 is greater than 5, the power of ten has been rounded up).

Question

What is the order of magnitude of the mass of Jupiter, 1.9×10^{27} kg?

Answer

Mass of Jupiter $\sim 10^{27}$ kg (since 1.9 is less than 5, the power of ten remains unchanged).

Question

What is the order of magnitude of the average lifetime of unstable 'sigma plus' particles, 0.7×10^{-10} s?

Answer

$$
\begin{aligned}
\text{Particle lifetime} &= 0.7 \times 10^{-10}\ \text{s} \\
&= 7 \times 10^{-11}\ \text{s} \\
&\sim 10^{(-11+1)}\ \text{s} \\
&\sim 10^{-10}\ \text{s}
\end{aligned}
$$

Since 7 is greater than 5, the power of ten must be rounded up

The phrase 'order of magnitude' is also quite commonly used to compare the sizes of things, e.g. a millimetre is three orders of magnitude smaller than a metre.

Worked example 2.4

To the nearest order of magnitude, how many times more massive is Jupiter than the Earth?

Answer

We had:

$$\text{mass of Jupiter} \sim 10^{27}\ \text{kg}$$

and

$$\text{mass of Earth} \sim 10^{25}\ \text{kg}$$

so

$$\frac{\text{mass of Jupiter}}{\text{mass of Earth}} \sim \frac{10^{27}}{10^{25}} \sim 10^{(27-25)} \sim 10^{2}$$

Jupiter is two orders of magnitude (i.e. roughly 100 times) more massive than the Earth.

Question 2.4

What is the order of magnitude of the following measurements?

(a) The distance between Pluto (the furthest planet in the Solar System) and the Sun: five thousand nine hundred million kilometres. Answer

(b) The diameter of the Sun, given that its radius is 6.97×10^7 m. Answer

(c) 2π. Answer

(d) The mass of a carbon dioxide molecule: 7.31×10^{-26} kg. Answer

Sophisticated instrumentation now allows scientists to measure across 40 orders of magnitude, as shown in Figure 2.2. If you turn back to Figure 1.2, you will see that the scale there is quite different to that in Figure 2.2. On the thermometer, the interval between marked points was always the same, with marked points at $-0.1, 0, 0.1, 0.2$, etc. In other words, each step from one division to the next on the scale represented the *addition or subtraction* of a fixed amount (0.1 in that case). This kind of scale is called linear. In Figure 2.2, on the other hand, each step involves *multiplication or division* by a fixed power of ten (10^2 in this particular case). As a result, the intervals between divisions are all different. This kind of scale is called logarithmic. The next question allows you to investigate some of the properties of this type of scale.

Question 2.5

Use information from Figure 2.2 to answer the following questions.

(a) What is the difference in value between: Answer

(i) the tick marks at 10^{-2} m and 10^{0} m;

(ii) the tick marks at 10^{0} m and 10^{2} m, and

(iii) the tick marks at 10^{2} m and 10^{4} m?

(b) Calculate to the nearest order of magnitude, how many times taller than a child is Mount Everest. Answer

(c) Calculate to the nearest order of magnitude, how many typical viruses laid end to end would cover the thickness of a piece of paper. (*Hint*: you may find it helpful to look back at Worked example 2.4.) Answer

2.3.1 Logarithmic scales in practice

In Figure 2.2, a logarithmic scale was used for the purposes of display, and the power of ten for the multiplying factor (10^2) was chosen because it was the one that best fitted the page. In drawing diagrams and graphs we are always free to choose the scale divisions. However, logarithmic scales are used in a number of fields to measure quantities that can vary over a very wide range. In such cases, an increase or decrease of one 'unit' always represents a ten-fold increase or decrease in the quantity measured. The following sections give two examples.

Sound waves

The *decibel* (symbol dB) is the unit used to measure the relative loudness of sounds. The 'intensity' of a sound is related to the square of the variation in pressure as the sound wave passes through the air, and the range of intensities that people can detect is enormous. The sound that just causes pain is 10^{12} times more intense than the sound that is just audible! To deal with this huge range, a logarithmic scale for loudness was devised, according to which every 10 dB (or '1 B') increase in sound level is equivalent to a 10-fold increase in intensity. The decibel is also a convenient measure because a sound level of 1 dB is just within the limit of human hearing, and a change of 1 dB is about the smallest difference in sound that the ear can detect. (See Figure 2.3.)

Earthquakes

The *Richter scale* describes the magnitude of earthquakes. An instrument called a seismometer is used to measure the maximum ground movement caused by the earthquake, and a correction factor is applied to this reading to allow for the distance of the seismometer from the site of the earthquake. Seismometers are very sensitive and can detect minute amounts of ground movement (they have to be shielded from the effects caused just by people walking near them), but some earthquakes can produce ground movements millions of times greater than the minimum detectable limit. To cope with this huge variation, the Richter scale is logarithmic: an increase of one unit on the scale implies a ten-fold increase in the maximum ground movement. A magnitude 2 earthquake can just be felt as a tremor. A magnitude 3 earthquake produces 10 times more ground motion than a magnitude 2 earthquake. Damage to buildings occurs at magnitudes in excess of 6. The three largest earthquakes ever recorded (in Portugal in 1775, in Columbia in 1905 and in Japan in 1933) each had a Richter magnitudes of 8.9.

Worked example 2.5

A whisper corresponds to a sound level of about 20 dB, and a shout to a level of about 80 dB. How much greater is the intensity of a shout compared to that of a whisper?

Answer

The increase in sound level is

$$80\ \text{dB} - 20\ \text{dB} = 60\ \text{dB}$$

This may be expressed as (10 dB + 10 dB + 10 dB + 10 dB + 10 dB + 10 dB), and *each* 10 dB increase corresponds to multiplying the intensity by 10.

So the intensity of a shout is $(10 \times 10 \times 10 \times 10 \times 10 \times 10) = 10^6$ times greater than a whisper!

Question 2.6 Answer

How much more ground movement is there in an earthquake measuring 7 on the Richter scale compared to one measuring 3?

The basis of logarithmic scales will be discussed in Chapter 7.

2.4 How precise are the measurements?

Scientists are always trying to get better and more reliable data. One way of getting a more precise measurement might be to switch to an instrument with a more finely divided scale. Figure 2.4 shows parts of two thermometers placed side by side to record the air temperature in a room.

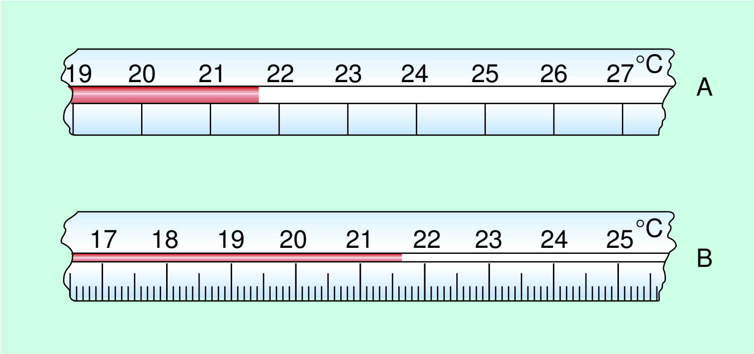


Figure 2.4: Parts of two thermometers A and B, measuring the air temperature in the same place.

The scale on thermometer A is quite coarse. The marked divisions represent integer numbers of degrees. On this scale we can see that the temperature is between 21 °C and 22 °C. I might estimate it as 21.7 °C, but somebody else could easily record it as 21.6 °C or 21.8 °C. So there is some uncertainty in the first decimal place, and certainly there is no way we could attempt to guess the temperature to two decimal places using this particular thermometer.

Thermometer B has a finer scale, with divisions marked every 0.1 °C. Now we can clearly see that the temperature is between 21.6 °C and 21.7 °C. I might read it as 21.63 °C, but a second person could plausibly read it as 21.61 °C or 21.65 °C. With this scale we are sure of the first decimal place but uncertain of the second.

When quoting the result of a measurement, you should never quote more digits than you can justify in terms of the uncertainty in the measurement. The number of significant figures in the value of a measured quantity is defined as the number of digits known with certainty plus one uncertain digit. With thermometer A we could be sure of the 21 (two digits), but were uncertain about the digit in the first decimal place, so we can quote a reading to three significant figures, as 21.7 °C (or 21.6 °C or 21.8 °C). With thermometer B it was the fourth digit that was uncertain, so we can quote our reading to four significant figures, as, for example, 21.64 °C.

Question 2.7 Answer

How many significant figures are quoted in each of the following quantities: 1221 m; 223.4 km; 1.487 km?

Question 2.7 emphasizes that significant figures mustn't be confused with the number of decimal places. After all, if you had measured the length of something as 13 mm, you wouldn't want the precision of your result to be changed just because you converted the measurement to centimetres. Whether you write 13 mm or 1.3 cm you are expressing the result of your measurement to two significant figures. Now suppose you convert to metres: 0.013 m. The uncertainty in your result still hasn't changed, so this shows that *leading zeroes in decimal numbers do not count as significant figures*. Scientific notation is helpful in this regard. Expressing the result as 1.3×10^{-2} m makes it very obvious that there are two significant figures.

Another circumstance in which one has to be careful about not using unjustified precision occurs when the results of measurements are used as the basis for calculations. Suppose we had measured the diameter of a circular pattern to two significant figures and obtained the result 3.3 cm. If we then needed to calculate the radius of the circle, it might be tempting simply to divide the diameter by 2 and say 'the radius of the pattern is 1.65 cm'. But 1.65 cm implies that the value is known to three significant figures! So we need to round off the figure in some way, to express the fact that the last significant digit in this particular case is the first digit after the decimal point. The usual rule for doing this is to leave the last significant digit unchanged if it would have been followed by a digit from 0 to 4, and to increase it by one if it would have been followed by a digit from 5 to 9. To two significant figures our circular pattern therefore has a radius of 1.7 cm. The issues involved in dealing with significant figures in more complex calculations are discussed in Chapter 3.

Scientific notation also shows up the need for care in dealing with very large numbers. The speed of light in a vacuum (the constant c in Einstein's equation $E = mc^2$ is, to six significant figures, 299 792 kilometres per second. Remembering the rounding rule, this can quite properly be written as 3×10^5 kilometres per second (one significant figure), or 3.00×10^5 kilometres per second (three significant figures). But it would be misleading to write it as 300 000 kilometres per second, because that could imply that all six digits are significant.

One of the advantages of using scientific notation is that it removes any ambiguity about whether zeroes at the *end* of a number are significant or are simply place markers. For example, if a length is measured to just one significant figure as 8 m, how should the equivalent value in centimetres be expressed? It would be ambiguous to write 800 cm, since that could imply the value is known to three significant figures. The only way out of this difficulty is to use scientific notation: writing 8×10^2 cm makes it clear that the quantity is known only to one significant figure, in line with the precision of the original measurement.

Question

If the speed of light through glass is quoted as 2.0×10^8 metres per second, how many significant figures are being given?

Answer

Final zeroes *are* significant, so the speed is being given to two significant figures.

Question

Neon gas makes up 0.0018% by volume of the air around us. How many significant figures are being given in this percentage?

Answer

Leading zeroes are *not* significant, so this value is also being given to two significant figures.

Worked example 2.6

The average diameter of Mars is 6762 km. What is this distance in metres, expressed to three significant figures?

Answer

The only way to express this quantity unambiguously to fewer than the four significant figures originally given is to use scientific notation.

$$\begin{aligned} 6762\ \text{km} &= 6.762 \times 10^3\ \text{km} \\ &= 6.762 \times 10^3 \times 10^3\ \text{m} \\ &= 6.762 \times 10^{(3+3)}\ \text{m} \\ &= 6.762 \times 10^6\ \text{m} \end{aligned}$$

The final digit is a 2, so no rounding up is required and the average diameter of Mars is 6.76×10^6 m to three significant figures.

Question 2.8

Express the following temperatures to two significant figures:

(a) -38.87 °C (the melting point of mercury, which has the unusual property for a metal of being liquid at room temperature); Answer

(b) -195.8 °C (the boiling point of nitrogen, i.e. the temperature above which it is a gas); Answer

(c) 1083.4 °C (the melting point of copper). Answer

In the following chapter and in your future studies of science generally, you will be doing lots of calculations with numbers in scientific notation, and will also be expected to quote your results to appropriate numbers of significant figures. Chapter 3 will discuss the efficient way to input scientific notation into your calculator, and how to interpret the results.

2.5 Learning outcomes for Chapter 2

After completing your work on this chapter you should be able to:

2.1 convert quantities expressed as integers or in decimal notation to scientific notation and vice versa;

2.2 use prefixes in association with the SI base units and convert between prefixes;

2.3 express a given quantity as an order of magnitude;

2.4 state the number of significant figures in any given quantity;

2.5 express a given quantity to any stipulated number of significant figures.

Calculating in Science

3

There comes a point in science when simply measuring is not enough and we need to *calculate* the value of a quantity from values for other quantities that have been measured previously. Take, for example, the piece of granite shown in Figure 3.1. We can measure the lengths of its sides and its mass. With a little calculation we can also find its volume, its density, and the speed at which seismic waves will pass through a rock of this type following an earthquake.

This chapter looks at several scientific calculations, and in the process considers the role of significant figures, scientific notation and estimating when calculating in science. In addition, it introduces unit conversions and the use of formulae and equations.

Figure 3.1: A specimen of granite.

3.1 Calculating area; thinking about units and significant figures

Suppose we want to find the area of the top of the granite specimen shown in Figure 3.1. The lengths of its sides, measured in centimetres, are shown in Figure 3.2, and the area of a rectangle is given by

$$\text{area of rectangle} = \text{length} \times \text{width}$$

Thus the area of the top of the granite is

$$\text{area} = 8.4 \text{ cm} \times 5.7 \text{ cm}$$

Multiplying the two numbers together gives 47.88. However, if given as a value for the area, this would be incomplete and incorrectly stated for two reasons.

1 No units have been given.

2 The values for length and width which we've used are each given to two significant figures, but 47.88 is to *four* significant figures. This is too many.

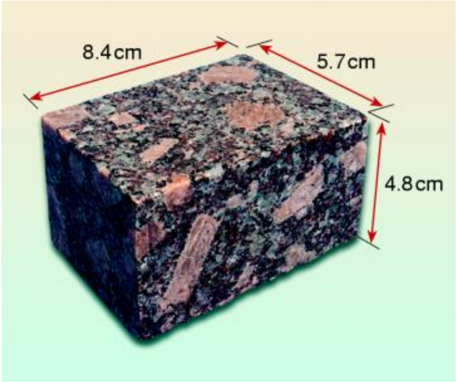


Figure 3.2: The lengths of the sides of the specimen of granite.

3.1.1 Units in calculations

The length and the width of the specimen of granite aren't just numbers, but physical quantities, with units. The area — the result of multiplying the length by the width — is a physical quantity too and it should also have units. The units which have been multiplied together are cm $\times$ cm, which can be written as $(\text{cm})^2$, or more commonly as cm^2. In fact any unit of length squared will be a unit of area. Conversely, a value given for area should *always* have units of $(\text{length})^2$.

All measurements should be given with appropriate units, and when performing calculations the units of the answer must always be consistent with the units of the quantities you input.

Care needs to be taken when multiplying together two lengths which have been measured in different units. Suppose, for instance, that we needed to find the area of a 1 cm by 4 m rectangle. Units of cm $\times$ m are meaningless; we need to convert the units to the same form before proceeding, and if in doubt it is best to convert to SI base units. Since 1 cm = 0.01 m, this gives an area of 0.01 m $\times$ 4 m = 0.04 m^2.

Question 3.1 Answer

Which of the following are units of area:

$(\text{inch})^2$; s^2; m^{-2}; cm^2; km^3; square miles?

Note: the symbols used for SI units are as given in Box 2.1.

3.1.2 Significant figures and rounding in calculations

It is not appropriate to quote answers to calculations to an unlimited number of significant figures. Suppose that, as part of a calculation, you were asked to divide 3.4 (known to two significant figures) by 2.34 (known to three significant figures). Entering $3.4 \div 2.34$ on most scientific calculators gives 1.452 991 453, but to quote a result to this number of significant figures would imply that you know the answer far more precisely than is really the case. The fact that 3.4 is quoted to two significant figures implies that the first digit is precisely known, but there is some uncertainty in the second digit; similarly the fact that 2.34 is quoted to three significant figures implies that there is some uncertainty in the third digit. Yet in giving the result as 1.452 991 453 we are claiming to be absolutely confident of the answer as far as 1.452 991 45, with just some uncertainty in the final digit. This is clearly nonsense!

The sensible number of significant figures to quote in any answer depends on a number of factors. However, in the absence of other considerations, a simple rule of thumb is useful:

When multiplying and dividing numbers, the number of significant figures in the result should be the same as in the measurement with the *fewest* significant figures.

Applying this rule of thumb, the answer to the calculation $3.4 \div 2.34$ should be given to two significant figures, i.e. as 1.5.

Similarly, the result of the multiplication $8.4\ \text{cm} \times 5.7\ \text{cm}$ (used in finding the area of the top of the granite specimen) should be given as $48\ \text{cm}^2$, again to two significant figures.

There are two points of caution to bear in mind when thinking about the appropriate number of significant figures in calculations.

Avoiding rounding errors

You should round your answer to an appropriate number of significant figures at the end of a calculation. However, be careful not to round too soon, as this may introduce unnecessary errors, known as rounding errors. As an example of the dangers of rounding errors, let's return to our previous example. We found that:

$$3.4 \div 2.34 = 1.452\,991\,453$$

Or, giving the answer to two significant figures:

$$3.4 \div 2.34 = 1.5$$

Suppose that we now need to multiply the answer by 5.9:

$$1.452\,991\,453 \times 5.9 = 8.572\,649\,573 = 8.6 \text{ to two significant figures}$$

However, using the intermediate answer as quoted to two significant figures gives

$$1.5 \times 5.9 = 8.85 = 8.9 \text{ to two significant figures}$$

Rounding too soon has resulted in an incorrect answer.

The use of scientific calculators enables us to work to a large number of significant figures and so to avoid rounding errors. If this is not possible, you should follow the following advice:

> Work to at least one more significant figure than is required in the final answer, and just round at the end of the whole calculation.

In our example, the final answer should be given to two significant figures, which means that we should work using the result of the first calculation to at least three significant figures (1.45).

$$1.45 \times 5.9 = 8.555 = 8.6 \text{ to two significant figures.}$$

Applying common sense!

Always bear in mind the real problem that you are solving, and apply common sense in deciding how to quote the answer. Particular care needs to be taken when the calculation involves numbers which are *exactly* known. A light-hearted example should illustrate this point.

Question

Suppose you have 7 apples to share between 4 children. How many apples does each child get?

Answer

Dividing the number of apples by the number of children gives

$$\frac{7}{4} = 1.75$$

If we were to assume that the number of apples and number of children were each quoted to one significant figure, we would round the answer to one significant figure too, i.e. to 2 apples. But we would then need eight apples, which is more than we've got. In reality there are *exactly* 4 children and 7 apples, so the number of significant figures need not bother us. Provided we have a knife, it is perfectly possible to give each child 1.75 ($1\frac{3}{4}$) apples.

Question 3.2

Do the following calculations and express your answers to an appropriate number of significant figures.

(a) $\dfrac{6.732}{1.51}$ Answer

(b) 2.0×2.5 Answer

(c) $\left(\dfrac{4.2}{3.1}\right)^2$ Answer

(d) What is the total mass of three 1.5 kg bags of flour? Answer

3.2 Calculating in scientific notation

In science it is very often necessary to do calculations using very large and very small numbers, and scientific notation can be a tremendous help in this.

3.2.1 Calculating in scientific notation without a calculator

Suppose we need to multiply 2.50×10^4 and 2.00×10^5. The commutative nature of multiplication is completely general, so it applies when multiplying two numbers written in scientific notation too. This means that $(2.50 \times 10^4) \times (2.00 \times 10^5)$ can be written as $(2.50 \times 2.00) \times (10^4 \times 10^5)$, i.e.

$$\begin{aligned}(2.50 \times 10^4) \times (2.00 \times 10^5) &= (2.50 \times 2.00) \times (10^4 \times 10^5) \\ &= 5.00 \times 10^{4+5} \\ &= 5.00 \times 10^9\end{aligned}$$

All of the rules for the manipulation of powers discussed in Chapter 1 can be applied to numbers written in scientific notation, but care needs to be taken to treat the decimal parts of the numbers (such as the 2.50 in 2.50×10^5) and the powers of ten separately. So, for example

$$\frac{2.50 \times 10^4}{2.00 \times 10^5} = \frac{2.50}{2.00} \times \frac{10^4}{10^5} = \frac{2.50}{2.00} \times 10^{4-5} = 1.25 \times 10^{-1}$$

and

$$\left(2.50 \times 10^5\right)^2 = 2.50^2 \times \left(10^5\right)^2 = 6.25 \times 10^{10}$$

Question 3.3

Evaluate the following without using a calculator, giving your answers in scientific notation.

(a) $(3.0 \times 10^{6}) \times (7.0 \times 10^{-2})$ Answer

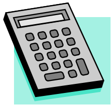
(b) $\dfrac{8 \times 10^{4}}{4 \times 10^{-1}}$ Answer

(c) $\dfrac{10^{4} \times (4 \times 10^{4})}{1 \times 10^{-5}}$ Answer

(d) $\left(3.00 \times 10^{8}\right)^{2}$ Answer

3.2.2 Using a calculator for scientific notation

In the rest of this chapter, and in your future studies of science generally, you will be doing many calculations with numbers in scientific notation, so it is very important that you know how to input them into your calculator efficiently and how to interpret the results.



First of all make sure that you can input numbers in scientific notation into your calculator.You can do this using the button you used to input powers in Section 1.3.1, but it is more straightforward to use the special button provided for entering scientific notation. This might be labelled as EXP, EE, E or EX, but there is considerable variation between calculators. Make sure that you can find the appropriate button on your calculator. Using a button of this sort is equivalent to typing the whole of '$\times 10$ to the power'. So, on a particular calculator, keying 2.5 EXP 12 enters the whole of 2.5×10^{12}.

In addition to being able to enter numbers in scientific notation into your calculator, it is important that you can understand your calculator display when it gives an answer in scientific notation.

Enter the number 2.5×10^{12} into your calculator and look at the display.

Again there is considerable variation from calculator to calculator, but it is likely that the display will be similar to one of those shown in Figure 3.3. The 12 at the right of the display is the power of ten, but notice that *the ten itself is frequently not displayed*. If your calculator is one of those which displays 2.5×10^{12} as shown in Figure 3.3e, then you will need to take particular care; this *does not* mean 2.5^{12} on this occasion. You should be careful not to copy down a number displayed in this way on your calculator as an answer to a question; this could cause confusion at a later stage.

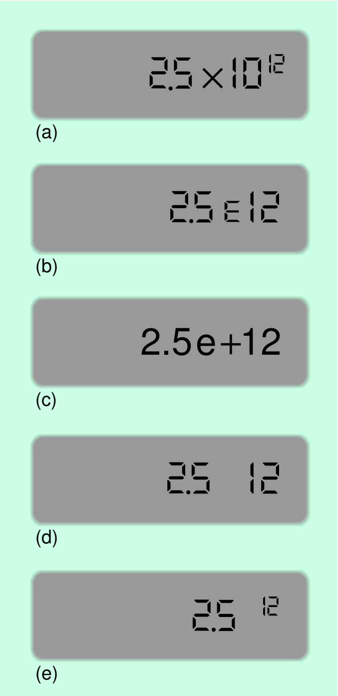


Figure 3.3: Examples of how various calculators would display the number 2.5×10^{12}

No matter how scientific notation is entered and displayed on your calculator or computer, when writing it on paper you should always use the form exemplified by 2.5×10^{12}.

To enter a number such as 5×10^{-16} into your calculator, you may need to use the button labelled something like +/− (as used in Section 1.1.3) in order to enter the negative exponent.

To enter a number such as 10^8 into your calculator using the scientific notation button, it can be helpful to remember that 10^8 is written as 1×10^8 in scientific notation, so you will need to key something like 1 EXP 8.

If you are at all unsure about using your calculator for calculations involving scientific notation, you should repeat Question 3.3, this time using your calculator.

Question 3.4 Answer

A square integrated circuit, used as the processor in a computer, has sides of length 9.78×10^{-3} m. Give its area in m^2 in scientific notation and to an appropriate number of significant figures.

3.3 Estimating answers

The first time I attempted Question 3.4, my calculator gave me the answer 95.6 m^2. This is incorrect (I'd forgotten to enter the power of ten). It is sensible to get into the habit of checking that the answer your calculator gives is reasonable, by estimating the likely answer. In the case of Question 3.4, the answer should be *approximately* $\left(1 \times 10^{-2}\ \mathrm{m}\right)^2$ which you can see (without using a calculator!) is $1 \times 10^{-4}\ \mathrm{m}^2$. So a calculator answer of 95.6 m^2 is clearly wrong.

In addition to being useful as a way of checking calculator answers, estimated answers are, in their own right, quite frequently all that is needed. Chapter 2 began with a comparison between the size of a bacterium and the size of a pinhead. We could use precise measuring instruments to find that the diameter of a particular bacterium is 1.69 μm (i.e. 1.69×10^{-6} m) and that the diameter of the head of a particular pin is 9.86×10^{-4} m. The diameter of the pinhead would then be

$$\frac{9.86 \times 10^{-4}\ \text{m}}{1.69 \times 10^{-6}\ \text{m}} = 5.83 \times 10^{2} \text{ times bigger than that of the bacterium.}$$

However, to get a feel for the relative sizes, we only really need to estimate the answer. If an estimate is all that is required, it is perfectly acceptable to work to one significant figure throughout (indeed, working to the nearest order of magnitude is sometimes sufficient) and since the final answer is only approximately known, the symbol '≈' (meaning 'approximately equal to') is used in place of an equals sign.

Worked example 3.1

Working to one significant figure throughout, estimate how many times bigger a pinhead of diameter 9.86×10^{-4} m is than a bacterium of diameter 1.69×10^{-6} m.

Answer

Diameter of pinhead $\approx 1 \times 10^{-3}$ m.
Diameter of bacterium $\approx 2 \times 10^{-6}$ m.

$$\begin{aligned}
\frac{\text{diameter of pinhead}}{\text{diameter of bacterium}} &\approx \frac{1 \times 10^{-3}\ \text{m}}{2 \times 10^{-6}\ \text{m}} \\
&\approx \frac{1}{2} \times \frac{10^{-3}}{10^{-6}} \\
&\approx 0.5 \times 10^{-3-(-6)} \\
&\approx 0.5 \times 10^{3} \\
&\approx 5 \times 10^{2}
\end{aligned}$$

So the diameter of the pinhead is approximately 500 times that of the bacterium.

It is important that you write out your mathematical calculations carefully, and one of the functions of the worked examples scattered throughout the course is to illustrate how to do this. There are three particular points to note from Worked example 3.1.

Taking care when writing maths

1 Note that the symbols = and ≈ mean ‘equals’ and ‘approximately equals’ and should *never* be used to mean ‘thus’ or ‘therefore’. It is acceptable to use the symbol ∴ for ‘therefore’; alternatively don’t be afraid to write *words* of explanation in your calculations.

2 It can make a calculation clearer if you align the = or ≈ symbols vertically, to indicate that the quantity on the left-hand side is equal to or approximately equal to each of the quantities on the right-hand side.

3 Note that the diameter of the bacterium and the pinhead each have metres (m) as their units, so when one diameter is divided by the other, the units cancel to leave a number with no units.

The handling of units in calculations is discussed further in Section 3.5.4.

Question 3.5 Answer

The average distance of the Earth from the Sun is 1.50×10^{11} m and the distance to the nearest star other than the Sun (Proxima Centauri) is 3.99×10^{16} m. Working to one significant figure throughout, estimate how many times further it is to Proxima Centauri than to the Sun.

3.4 Unit conversions

In calculating the area of the top of the granite specimen earlier in this chapter, we measured the length of the sides in centimetres and hence calculated the area in cm^2. If we had wanted the area in the SI units of m^2 we could have converted the lengths from centimetres to metres before starting the calculation. We would then have had

$$\text{area} = (8.4 \times 10^{-2}\ \text{m}) \times (5.7 \times 10^{-2}\ \text{m}) = 4.8 \times 10^{-3}\ \text{m}^2$$

It is best, whenever possible, to convert all units to SI units before starting on a calculation.

Unfortunately it is not always possible to convert units before commencing a calculation; sometimes you will be given an area in, say, cm^2, without knowing how the area was calculated, and you will need to convert this to an area in m^2. This section discusses this, as well as some more complex unit conversions.

3.4.1 Converting units of area

Let's start with an example which is relatively easy to visualize. Suppose we want to know how many mm^2 there are in a cm^2. There are 10 millimetres in a centimetre, so each side of the square centimetre in Figure 3.4 measures either 1 cm or 10 mm. To find the area, we need to multiply the length by the width. Working in centimetres gives

$$\text{area} = 1\text{ cm} \times 1\text{ cm} = (1\text{ cm})^2 = 1^2\text{ cm}^2 = 1\text{ cm}^2$$

Working in millimetres gives

$$\text{area} = 10\text{ mm} \times 10\text{ mm} = (10\text{ mm})^2 = 10^2\text{ mm}^2 = 100\text{ mm}^2$$

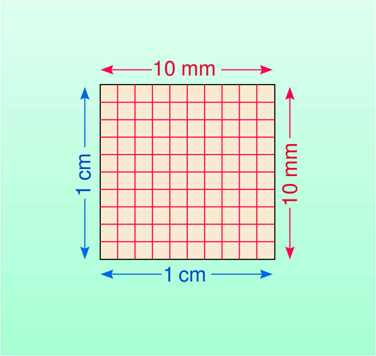


Figure 3.4: A square centimetre (not to scale)

Thus $1\text{ cm}^2 = 100\text{ mm}^2$ and $1\text{ mm}^2 = \dfrac{1}{100}\text{ cm}^2$.

If we want to convert from cm^2 to mm^2 we need to multiply by 100; if we want to convert from mm^2 to cm^2 we need to divide by 100.

Figure 3.5 illustrates another example which is a little harder to visualize. Each side of the square measures either 1 km or 1000 m (10^3 m). Working in kilometres gives

$$\text{area} = 1\text{ km} \times 1\text{ km} = (1\text{ km})^2 = 1^2\text{ km}^2 = 1\text{ km}^2$$

Working in metres gives

$$\text{area} = 10^3\text{ m} \times 10^3\text{ m} = \left(10^3\text{ m}\right)^2 = \left(10^3\right)^2\text{ m}^2 = 10^6\text{ m}^2$$

Thus $1\text{ km}^2 = 10^6\text{ m}^2$ and $1\text{ m}^2 = \dfrac{1}{10^6}\text{ km}^2$.

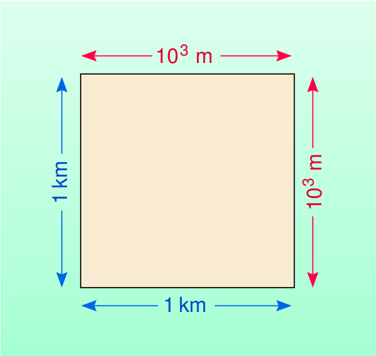


Figure 3.5: A square kilometre

To convert from km^2 to m^2 we need to multiply by 10^6; to convert from m^2 to km^2 we need to divide by 10^6.

The number by which we need to divide or multiply to convert from one unit to another is known as the ‘conversion factor’. In general, to convert between units of area we need to *square* the conversion factor which we would use to convert corresponding lengths.

As a final example consider a conversion between km^2 and mm^2.

There are 10^3 millimetres in a metre and 10^3 metres in a kilometre, so there are 10^6 millimetres in a kilometre as illustrated in Figure 3.6.

To convert from kilometres to millimetres we need to multiply by 10^6; however to convert from km^2 to mm^2 we need to multiply by $\left(10^6\right)^2$, i.e. 10^{12}.

Similarly, to convert from mm^2 to km^2 we need to divide by $\left(10^6\right)^2$, i.e. 10^{12}.

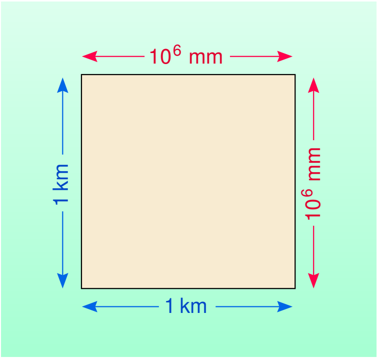


Figure 3.6: A square kilometre

Question 3.6

A desk has an area of $1.04\ m^2$. Express this area in:

(a) cm^2 Answer

(b) μm^2 Answer

(c) km^2 Answer

3.4.2 Converting units of volume

The volume of the piece of granite shown in Figure 3.2 is given by

$$\text{volume} = \text{length} \times \text{width} \times \text{height}$$

The lengths of the sides are 8.4 cm, 5.7 cm and 4.8 cm, so

$$\begin{aligned}\text{volume} &= 8.4\ \text{cm} \times 5.7\ \text{cm} \times 4.8\ \text{cm}\\ &= 2.3 \times 10^2\ \text{cm}^3 \text{ to two significant figures.}\end{aligned}$$

Note that the units which have been multiplied together are cm × cm × cm, so in this case the units of volume are cm^3. A value given for volume should *always* have units equivalent to those used for $(\text{length})^3$, and if we had converted the lengths of the sides to metres before doing the calculation, we would have obtained a value for volume in m^3:

$$\begin{aligned}\text{volume} &= (8.4 \times 10^{-2}\ \text{m}) \times (5.7 \times 10^{-2}\ \text{m}) \times (4.8 \times 10^{-2}\ \text{m})\\ &= 2.3 \times 10^{-4}\ \text{m}^3 \text{ to two significant figures.}\end{aligned}$$

The method for converting between different units of volume is a direct extension of the method for converting between different units of area. Suppose we want to know how many mm^3 there are in a cm^3.

There are 10 mm in 1 cm, so each side of the cubic centimetre in Figure 3.7 measures either 1 cm or 10 mm. The volume can be written as either 1 cm^3 or 10^3 mm^3. Thus 1 $\text{cm}^3 = 10^3$ mm^3 and 1 $\text{mm}^3 = \dfrac{1}{10^3}$ cm^3. To convert from cm^3 to mm^3 we need to multiply by 10^3; to convert from mm^3 to cm^3 we need to divide by 10^3.

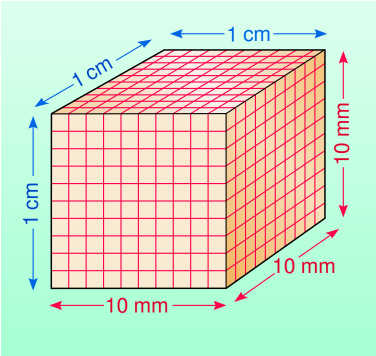


Figure 3.7: A cubic centimetre (not to scale).

In general, to convert between units of volume we need to *cube* the conversion factor that we would use to convert corresponding lengths.

We can convert a volume of 2.3×10^2 cm^3 into m^3 simply by saying that there are 10^2 cm in 1 m; hence there are $\left(10^2\right)^3$ cm^3 in 1 m^3, so

$$1\ \text{cm}^3 = \frac{1}{(10^2)^3}\ \text{m}^3$$

and

$$\begin{aligned} 2.3 \times 10^2\ \text{cm}^3 &= \frac{2.3 \times 10^2}{(10^2)^3}\ \text{m}^3 \\ &= 2.3 \times 10^{-4}\ \text{m}^3 \end{aligned}$$

This value is, of course, the same as the one we obtained from first principles!

The prefix ‘deci’ meaning one tenth was introduced in Section 2.2, thus 1 decimetre (dm) is one tenth of a metre. The cubic decimetre (dm^3) is sometimes used as a unit of volume. The litre (l) (also introduced in Chapter 2) was defined in 1901 as the volume of a kilogram of water at 4 °C, under standard atmospheric pressure. This volume turns out to be 1.000 28 dm^3, and since 1969 a litre has been *defined* to be 1 dm^3.

Worked example 3.2

Convert a volume of 1 dm^3 to: (a) cm^3 (b) m^3

Answer

(a) 1 m = 10 dm and 1 m = 100 cm so 1 dm = 10 cm.

Thus 1 $dm^3 = 10^3$ cm^3.

(b) 1 m = 10 dm

Thus 1 $m^3 = 10^3$ dm^3

and $1\ dm^3 = \dfrac{1}{10^3}\ m^3 = 10^{-3}\ m^3$.

{Thus 1 dm^3 (i.e. 1 litre) is a thousand times bigger than a cubic centimetre and a thousand times smaller than a cubic metre. You may already have been aware that 1 litre = 1000 cm^3. Thus 1 ml = 1 cm^3. }

Figure 3.8 is a summary of unit conversions for length, area and volume, but you should try to remember the general principles involved rather than memorizing individual conversion factors.

Question 3.7

Express each of the following volumes in scientific notation in m^3:

(a) the volume of the planet Mars, which is 1.64×10^{11} km^3; Answer

(b) the volume of a ball bearing, which is 16 mm^3. Answer

3.4.3 Converting units of distance, time and speed

You were introduced in Box 2.1 to the metre as the base unit of distance or length and to the second as the base unit of time. The average speed with which an object moves is the total distance travelled divided by the total time taken, so when Marion Jones won the women's 100-metre final at the 2000 Sydney Olympics in 10.75 s, her average speed was

$$\text{average speed} = \frac{100.0\ \text{m}}{10.75\ \text{s}} = 9.302\ \text{m}\,\text{s}^{-1}$$

Similarly, if a girl grows a total of 116 cm in 12.5 years, her average rate of growth is

$$\text{growth rate} = \frac{116\ \text{cm}}{12.5\ \text{years}} = 9.28\ \text{cm}\,\text{year}^{-1}$$

Note that it is appropriate to give the answer to the first example to four significant figures (assuming that the length of the running track was known to at least four significant figures). Also note the way in which the units have been written in both examples.

The notation of negative exponents, which we have used to represent numbers like $1/2^3$ as 2^{-3} and $1/10^8$ as 10^{-8}, can also be used for units. So $1/\mathrm{s}$ can be written as s^{-1}, m/s can be written as $\mathrm{m\,s}^{-1}$ and cm/year can be written as $\mathrm{cm\,year}^{-1}$.

The SI unit of speed is $\mathrm{m\,s}^{-1}$ and this is usually said as 'metres seconds to the minus one'. Although $\mathrm{m\,s}^{-1}$ is the correct scientific way of writing the unit, it is sometimes written as m/s, and quite frequently said as 'metres per second', even when written as $\mathrm{m\,s}^{-1}$. The '/' for per is quite commonly used in other units too.

Many things move and/or grow in the world around us, and it is useful to compare different values for speed or rate of growth. Different speeds are frequently measured in different units, so in order to be able to compare like with like it is necessary to convert between different units for distance, time and speed. Box 3.1 considers various examples of speed and growth, and the text immediately following the box looks at ways of converting one unit to another.

Box 3.1 How fast?

Light (and other forms of radiation such as X-rays and radio waves) travels in a vacuum with a constant speed of $3.00 \times 10^8\ \mathrm{m\,s^{-1}}$. It is currently believed that nothing can travel faster than this.

Towards the opposite extreme are stalactites and stalagmites, which grow just fractions of a millimetre each year. A typical growth rate is $0.1\ \mathrm{mm\,year^{-1}}$. Stalactites form when water drips from the roof of an underground cave, depositing calcite (frequently from the limestone in the rock above the cave) in an icicle shaped formation as it does so. Stalagmites form as the water drips onto the floor of the cave, depositing further calcite.

Figure 3.9: The Saskatchewan Glacier, Banff National Park, Canada.

It is not normally possible to detect the motion of a glacier by eye, but there is considerable variation in the speed with which they move. The Franz Josef Glacier in New Zealand is particularly fast moving, with an average speed of about $1.5\ \mathrm{m\,day^{-1}}$. The speed of the Saskatchewan Glacier in Canada (Figure 3.9) is rather more typical, at about $12\ \mathrm{cm\,day^{-1}}$.

In addition to geological processes such as glacier flow and stalactite formation, the theory of plate tectonics tells us that the surface of the Earth is itself moving.

The Earth's surface is thought to comprise seven major tectonic plates and numerous smaller ones, each only about 100 km thick but mostly thousands of kilometres in width. Evidence, including evidence from sea-floor spreading (to be discussed in Chapter 5) indicates that plates move relative to one another with speeds between about $10\ \mathrm{km\,Ma^{-1}}$ and $100\ \mathrm{km\,Ma^{-1}}$ (where Ma is the abbreviation for a million years, as discussed in Section 2.2).

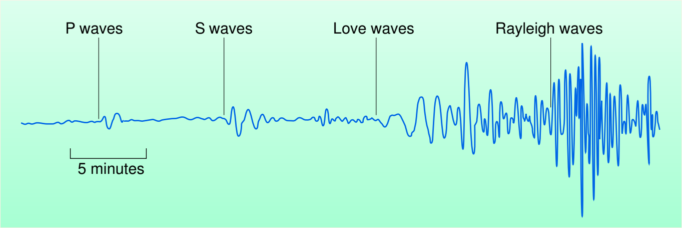


Figure 3.10: A seismogram (the printout from a seismometer) showing the arrival of P waves, S waves, Love waves and Rayleigh waves from a distant earthquake. Elapsed time increases from left to right.

Earthquakes and volcanoes occur all over the Earth, but they are more common close to the boundaries of tectonic plates than elsewhere. Following an earthquake, seismic waves (the word ‘seismic’ is from the Greek for ‘shaking’) travel out from the centre of the quake and are recorded by seismometers at various locations. There are several different types of seismic waves, including P waves, S waves, Love waves and Rayleigh waves, each travelling at different speeds (and sometimes also by different routes), so reaching a given seismometer at different times (see Figure 3.10). P waves travel fastest, with an average speed of about $5.6\ \mathrm{km\,s^{-1}}$ in rocks close to the Earth’s surface, so reach the seismometer first (the name P wave was originally an abbreviation for primary wave). S waves (S for secondary) travel with an average speed of about $3.4\ \mathrm{km\,s^{-1}}$ in rocks close to the Earth’s surface.

Perhaps the most dangerous sort of volcanic eruption is one that leads to a high-speed pyroclastic flow (a mixture of rock fragments and gases, moving as a fluid) away from the volcano. Pyroclastic flows are particularly destructive both because of their high temperatures (typically between 200 °C and 700 °C) and the high speed at which they travel (up to about 100 km hour^{-1}).

The speeds given so far have related to processes on the Earth, but remember that the Earth itself is moving too! The rotation of the Earth on its axis leads to a movement of up to 0.5 km s^{-1} at the surface. In addition, the Earth is orbiting the Sun at about 30 km s^{-1} and the entire Solar System is moving around the centre of the galaxy at about 250 km s^{-1}.

To convert from one unit of speed to another, we may need to convert both the unit of distance and the unit of time. To start with, let's consider the rather more straightforward case when we only have to convert the unit of distance, for example in converting from mm s^{-1} to m s^{-1}.

We know that 1 m = 10^3 mm

so $1\text{ mm} = \dfrac{1}{10^3}\text{ m} = 1 \times 10^{-3}\text{ m}$

We can therefore say straight away that $1\text{ mm s}^{-1} = 1 \times 10^{-3}\text{ m s}^{-1}$

We have simply applied the same conversion factor as in converting from mm to m. Note that the answer makes sense: it is reasonable to expect that the numerical value of a speed in m s^{-1} will be smaller than the same speed when given in mm s^{-1}.

Worked example 3.3

Convert the speed of the Earth as it orbits the Sun (given above as 30 $\mathrm{km\,s^{-1}}$) into a value in $\mathrm{m\,s^{-1}}$.

Answer

$1\ \mathrm{km} = 1 \times 10^3\ \mathrm{m}$

So

$$1\ \mathrm{km\,s^{-1}} = 1 \times 10^3\ \mathrm{m\,s^{-1}}$$

$$\begin{aligned} 30\ \mathrm{km\,s^{-1}} &= 30 \times 10^3\ \mathrm{m\,s^{-1}} \\ &= 3.0 \times 10^4\ \mathrm{m\,s^{-1}} \text{ in scientific notation.} \end{aligned}$$

The Earth orbits the Sun with a speed of about $3.0 \times 10^4\ \mathrm{m\,s^{-1}}$. Again the answer makes sense: it is reasonable to expect that the numerical value of a speed in $\mathrm{m\,s^{-1}}$ will be larger than the same speed when given in $\mathrm{km\,s^{-1}}$.

Next let's consider what happens when we need to convert only the time part of units of speed, for instance in converting from $\mathrm{km\,hour^{-1}}$ to $\mathrm{km\,s^{-1}}$.

We know that there are 60 minutes in an hour and 60 seconds in a minute, so

$$1\ \text{hour} = 60 \times 60\ \mathrm{s} = 3600\ \mathrm{s}$$

However, in this case we don't want to convert from hours to seconds, but rather from kilometres *per hour* to kilometres *per second*. The way forward comes in recognizing that the word 'per' and the use of negative exponents in hour^{-1} and s^{-1} indicate division. So to convert from hour^{-1} to s^{-1} (or from $\mathrm{km\,hour}^{-1}$ to $\mathrm{km\,s}^{-1}$) we need to find the conversion factor from hours to seconds and then *divide* by it.

$$\begin{aligned}&1 \text{ hour} = 3600 \text{ s}\\ &\text{so } 1\ \mathrm{km\,hour}^{-1} = \frac{1}{3600}\ \mathrm{km\,s}^{-1}\end{aligned}$$

In deciding whether to divide or multiply by a particular conversion factor, common sense can also come to our aid. It is reasonable to expect that a speed quoted in $\mathrm{km\,s}^{-1}$ will be *smaller* than the same speed when quoted in $\mathrm{km\,hour}^{-1}$, so it is reasonable to *divide* by the 3600 on this occasion.

Worked example 3.4

Two tectonic plates are moving apart at an average rate of $35\ \mathrm{km\,Ma^{-1}}$. Convert this to a value in $\mathrm{km\,year^{-1}}$.

Answer

We know that

$$1\ \mathrm{Ma} = 10^6\ \mathrm{years}$$

so

$$1\ \mathrm{km\,Ma^{-1}} = \frac{1}{10^6}\ \mathrm{km\,year^{-1}}$$

and therefore

$$\begin{aligned}35\ \mathrm{km\,Ma^{-1}} &= \frac{35}{10^6}\ \mathrm{km\,year^{-1}}\\ &= 3.5 \times 10^{-5}\ \mathrm{km\,year^{-1}}\ \text{in scientific notation.}\end{aligned}$$

The plates are moving apart at an average rate of $3.5 \times 10^{-5}\ \mathrm{km\,year^{-1}}$.

This answer is reasonable: you would expect the rate of separation quoted in $\mathrm{km\,year^{-1}}$ to be smaller than the same rate quoted in $\mathrm{km\,Ma^{-1}}$.

Question 3.8

Convert the average speed of the Saskatchewan Glacier (12 cm day^{-1}) to a value in:

(a) m day^{-1} Answer

(b) cm s^{-1} Answer

Finally we need to consider conversions for speed in which both the units of distance and the units of time have to be converted. This is simply a combination of the techniques illustrated in Worked examples 3.3 and 3.4. Suppose we want to convert from km hour^{-1} to m s^{-1}.

$$1 \text{ km} = 10^3 \text{ m}$$
$$1 \text{ hour} = 3600 \text{ s}$$

To convert from km hour^{-1} to m s^{-1}, we need to *multiply* by 10^3 (to convert the km to m) and *divide* by 3600 (to convert the hour^{-1} to s^{-1}):

$$1 \text{ km hour}^{-1} = \frac{10^3}{3600} \text{ m s}^{-1} = 0.278 \text{ m s}^{-1} \text{ to three significant figures.}$$

Worked example 3.5

Convert the average speed of separation of the tectonic plates discussed in Worked example 3.4 (35 km Ma^{-1}) to a value in mm $year^{-1}$.

Answer

1 km = 10^3 m and 1 m = 10^3 mm, so 1 km = 10^6 mm

1 Ma = 10^6 year

To convert from km Ma^{-1} to mm $year^{-1}$, we need to *multiply* by 10^6 (to convert the km to mm) and *divide* by 10^6 (to convert the Ma^{-1} to $year^{-1}$.

$$1\ \text{km Ma}^{-1} = \frac{10^6}{10^6}\ \text{mm year}^{-1} = 1\ \text{mm year}^{-1}$$

Thus a speed given in km Ma^{-1} is numerically equal to one given in mm $year^{-1}$. The plates are moving apart at a 35 mm $year^{-1}$. This is similar to the rate at which human fingernails grow and is easier to imagine than is 35 km Ma^{-1}.

Question 3.9

Convert each of the following to values in $\mathrm{m\,s^{-1}}$ and then compare them.

(a) A stalactite growth rate of $0.1\ \mathrm{mm\,year^{-1}}$. Answer

(b) The average speed of the Saskatchewan Glacier ($12\ \mathrm{cm\,day^{-1}}$). Answer

(c) The speed of separation of the tectonic plates discussed in Worked examples 3.4 and 3.5 ($35\ \mathrm{km\,Ma^{-1}}$). Answer

(*Note*: for the purposes of this question, consider 1 year to be 365 days long.)

3.4.4 Concentration and density; more unit conversions

Methods for converting units for physical quantities, such as concentration and density, follow directly from the discussion in the previous sections.

Box 3.2 Concentration

The concentration of a solution is a term used as a measure of how much of a certain substance the solution contains, relative to the solution's total volume. For example, we may want to know how much sugar has been dissolved in water to give one litre of syrup.

The amount of the substance can be measured in moles, in which case the concentration will have units of $\mathrm{mol\,l^{-1}}$ or $\mathrm{mol\,dm^{-3}}$. Alternatively, the amount can be measured by mass, in kg, g, mg, etc., leading to units for concentration of $\mathrm{kg\,dm^{-3}}$, $\mathrm{g\,m^{-3}}$, or $\mathrm{mg\,l^{-1}}$, and so on.

The World Health Organization (WHO) sets limits for safe concentrations of various impurities in water, for example, the limit for the concentration of nitrates in water is currently $50\ \mathrm{mg\,l^{-1}}$. This means that there should be no more than 50 mg of nitrate in each litre ($\mathrm{dm^3}$) of water.

To convert a concentration from, say, $\mathrm{mg\,l^{-1}}$ to $\mathrm{\mu g\,ml^{-1}}$ you need to follow a very similar procedure to the one introduced in Section 3.4.3, as the following worked example shows.

It is very easy to confuse the letter ‘l’, used as the symbol for litres, with the number 1. Take care!

Worked example 3.6

Convert 50 $\mathrm{mg\,l^{-1}}$ (the World Health Organization’s limit for the concentration of nitrates in water) to a value in $\mathrm{\mu g\,ml^{-1}}$.

Answer

We can easily write down the conversion factors for mg to μg and from litres to ml.

$$1\ \mathrm{mg} = 10^3\ \mathrm{\mu g}$$

$$1\ \text{litre} = 1\ \mathrm{l} = 10^3\ \mathrm{ml}$$

So to convert from $\mathrm{mg\,l^{-1}}$ to $\mathrm{\mu g\,ml^{-1}}$, we need to *multiply* by 10^3 (to convert the mg to μg) and *divide* by 10^3 (to convert the $\mathrm{l^{-1}}$ to $\mathrm{ml^{-1}}$).

$$1\ \mathrm{mg\,l^{-1}} = \frac{10^3}{10^3}\ \mathrm{\mu g\,ml^{-1}} = 1\ \mathrm{\mu g\,ml^{-1}}$$

Thus a concentration given in $\mathrm{mg\,l^{-1}}$ is numerically equal to one given in $\mathrm{\mu g\,ml^{-1}}$, in particular 50 $\mathrm{mg\,l^{-1}}$ = 50 $\mathrm{\mu g\,ml^{-1}}$.

Box 3.3 Density

The density of a piece of material is found by dividing its mass by its volume. In other words

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

If mass is measured in kg and volume is in m^3, then it follows that the unit of density will be kg/m^3 (said as 'kilograms per metre cubed') or, written in the form favoured in this course, $kg\,m^{-3}$ (said as 'kilograms metres to the minus three').

The density of pure water is $1 \times 10^3\ kg\,m^{-3}$; materials with a density greater than this (such as steel of density $7.8 \times 10^3\ kg\,m^{-3}$) will sink in water whereas materials of lower density (such as wood from an oak tree, density $6.5 \times 10^2\ kg\,m^{-3}$) will float.

If mass is measured in g and the volume is in cm^3, then the unit of density will be $g\,cm^{-3}$. Note that $g\,cm^{-3}$ is not an SI unit, but it is nevertheless quite frequently used.

Question

The specimen of granite shown in Figure 3.2 has a mass of 6.20×10^2 g. Calculate the density of the granite in $\mathrm{g\,cm^{-3}}$.

Answer

The volume of the specimen = 8.4 cm × 5.7 cm × 4.8 cm, so

$$\begin{aligned}
\text{density} &= \frac{\text{mass}}{\text{volume}} \\
&= \frac{6.20 \times 10^2\ \text{g}}{8.4\ \text{cm} \times 5.7\ \text{cm} \times 4.8\ \text{cm}} \\
&= 2.6977\ \mathrm{g\,cm^{-3}} \\
&= 2.7\ \mathrm{g\,cm^{-3}}\ \text{to two significant figures.}
\end{aligned}$$

Note that it was not necessary actually to calculate a value for volume before completing the calculation of density. If you had used the value for volume calculated at the beginning of Section 3.4.2, you would have obtained

$$\text{density} = \frac{\text{mass}}{\text{volume}} = \frac{6.20 \times 10^2\ \text{g}}{2.3 \times 10^2\ \text{cm}^3} = 2.7\ \mathrm{g\,cm^{-3}}$$

but you would have risked introducing rounding errors.

The final worked example in this section converts the units of the density of the granite specimen from $\mathrm{g\,cm^{-3}}$ to $\mathrm{kg\,m^{-3}}$, using a method which is a combination of the techniques taught throughout Section 3.4. You can convert units of concentration such as $\mathrm{mg\,dm^{-3}}$ to $\mathrm{g\,m^{-3}}$ in a similar way.

Worked example 3.7

Convert $2.7\ \mathrm{g\,cm^{-3}}$ (the density of the specimen of granite shown in Figures 3.1 and 3.2) to a value in the SI units of $\mathrm{kg\,m^{-3}}$.

Answer

$1\ \mathrm{kg} = 10^3\ \mathrm{g}$, so $1\ \mathrm{g} = \frac{1}{10^3}\ \mathrm{kg} = 10^{-3}\ \mathrm{kg}$

$1\ \mathrm{m} = 10^2\ \mathrm{cm}$, so $1\ \mathrm{m^3} = \left(10^2\right)^3\ \mathrm{cm^3} = 10^6\ \mathrm{cm^3}$ (from Section 3.4.2)

so $1\ \mathrm{cm^3} = \frac{1}{10^6}\ \mathrm{m^3} = 10^{-6}\ \mathrm{m^3}$

To convert from $\mathrm{g\,cm^{-3}}$ to $\mathrm{kg\,m^{-3}}$ we need to *multiply* by 10^{-3} (to convert the g to kg) and *divide* by 10^{-6} (to convert the $\mathrm{cm^{-3}}$ to $\mathrm{m^{-3}}$).

$$1\ \mathrm{g\,cm^{-3}} = \frac{10^{-3}}{10^{-6}}\ \mathrm{kg\,m^{-3}} = 10^{-3-(-6)}\ \mathrm{kg\,m^{-3}} = 10^3\ \mathrm{kg\,m^{-3}}$$

Thus $2.7\ \mathrm{g\,cm^{-3}} = 2.7 \times 10^3\ \mathrm{kg\,m^{-3}}$.

The specimen of granite has a density of $2.7 \times 10^3\ \mathrm{kg\,m^{-3}}$.

You may have already known that you need to multiply by 1000 in order to convert from units of g cm^{-3} to units of kg m^{-3}, but as was the case with the unit conversions for area and volume, it is better to consider general principles rather than trying to memorize conversion factors.

Question 3.10

The World Health Organization reduced its maximum recommended concentration for arsenic in drinking water from $50\ \mu\text{g l}^{-1}$ to $10\ \mu\text{g l}^{-1}$ in 1999. Convert $10\ \mu\text{g l}^{-1}$ to a value in:

(a) $\mu\text{g ml}^{-1}$ Answer

(b) mg dm^{-3} Answer

(c) g m^{-3} Answer

3.5 An introduction to symbols, equations and formulae

To progress further in our exploration of ways of calculating in science, we need to enter the world of symbols, equations and formulae. The word 'algebra' is used to describe the process of using symbols, usually letters, to represent quantities and the relationships between them. Algebra is a powerful shorthand that enables us to describe the relationships between physical quantities briefly and precisely, without having to know their numerical values. Some people consider algebra to be a beautiful thing: others are filled with terror by the very word. This course may not convince you of algebra's beauty, but it should at least illustrate its usefulness and give you an opportunity to learn and practise new techniques or revise old ones.

Chapter 4 is devoted to algebraic techniques such as simplifying, rearranging, and combining equations. The remainder of Chapter 3 simply introduces the language of algebra by looking at a few equations very carefully, and substituting values into them.

The word equation is used for an expression containing an equals sign. The quantities under consideration may be described in words, for example

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

in which case the equation is known as a 'word equation', or represented by symbols, for example

$$\rho = \frac{m}{V}$$

but the important thing to remember is that what is written on the left-hand side of the '=' sign must *always* be equal to what is written on the right-hand side. Thus, as explained in *Taking care when writing maths* in Section 3.3, you should never use '=' as a shorthand for anything other than 'equals'.

The word formula is used in mathematics to mean a rule expressed in algebraic symbols. Thus $\rho = \frac{m}{V}$ is a formula which tells you that the density ρ of a substance can be obtained by dividing the mass, m, of a sample of the substance by the volume, V, of the sample. Strictly speaking, not all equations are formulae, but the words tend to be used interchangeably.

3.5.1 What do the symbols mean?

Mathematics textbooks teaching algebra frequently contain page after page of equations of the form:

$$x + 3 = 8 \tag{3.1}$$

and

$$y = x + 5 \tag{3.2}$$

In Equation 3.1, x can only have one value, i.e. it is a constant. In this case x has the value 5. In Equation 3.2, x and y are *variables* which can each take an infinite number of values, but y will always be 5 greater than x. The values (of x and y, etc.)

which satisfy a particular equation are known as solutions and if you are asked to solve an equation you need to look for solutions.

In both Equation 3.1 and Equation 3.2, x and y represent pure *numbers*. Equations in science are often rather different. Rather than representing pure numbers, the symbols usually represent physical quantities and will therefore have *units* attached.

3.5.2 Which symbols are used

Box 3.4 contains a range of scientific formulae in common use, along with a brief explanation of the meaning of each symbol used. Have a quick at these equations now, but don't worry about their details; you are not expected to learn them or to understand the meanings of the scientific terms introduced. The equations in the boxes will be used as examples throughout the rest of this chapter, and have been numbered for ease of reference.

The symbol chosen to represent something is often the first letter of the quantity in question, e.g. m for mass, t for time and l for length, but it isn't always so simple. Greek letters are also frequently used as symbols e.g. λ (lambda) for wavelength in Equation 3.13 and ρ (rho) for density in Equations 3.9, 3.10 and 3.11. A list of Greek letters and their pronunciation is given in the Table 3.1 and you will soon become familiar with those that are commonly used. In a sense it doesn't matter which symbol you use to represent a quantity, since the symbol is only an arbitrarily chosen label. For instance, Einstein's famous equation (Equation 3.7) is usually written as $E = mc^2$, but the equation could equally well be written using any symbols you wanted to use, e.g. $p = qr^2$, provided you also made it clear that p was used to represent energy, q was used to represent mass and r was used to represent the speed of light. However, the use of conventional symbols, such as E for energy, saves scientists a lot of time in explaining their shorthand. *Maths for Science* follows convention as far as possible in its use of symbols. Sometimes the reason for the choice of symbol will be obvious but unfortunately this is not always the case.

Sometimes a subscript is used alongside a symbol in order to make its meaning more specific, as in v_{i}, v_{f} and v_{av} used in Equation 3.15 to mean initial, final, and average speed, and a_x in Equations 3.16 and 3.17 used to mean acceleration along the x-axis. Note that although a_x, for example, uses two letters, it represents a single physical entity; note also that a_x is *not* the same as ax. The symbol Δ (the Greek upper case delta) is frequently used to represent the change in a quantity, so ΔT in Equation 3.14 means a change in temperature T; again a *single* physical entity is represented by *two* letters.

A few letters have more than one conventional meaning, for example c in Equation 3.7 represents the speed of light, but in Equation 3.14 the same letter represents specific heat capacity. Other letters have two meanings but lower case is conventionally used for one meaning and upper case for the other, for example v for speed and V for volume or t for time and T for temperature. Care needs to be taken, but the intended meaning should be clear from the context.

Unfortunately some Greek letters look rather like everyday English ones; for example ρ (rho), used for density, can look rather like the English lower case p. Some textbooks use lower case p for pressure (this course uses capital P) and Equation 3.11 ($P = \rho g h$) can then appear to have the same quantity on both the left- and right-hand sides of the equals sign, especially when written out by hand. In reality, this formula has *pressure* on the left-hand side and *density* (and other things) on the right-hand side. A similar confusion can arise because the letter l can look like the number 1.

A final possible source of confusion stems from the fact that the same letter may sometimes be used to represent both a physical quantity and a unit of measurement. For example, an object with a mass of 6 kilograms and a length of 2 metres might be described by the relationships $m = 6$ kg, $l = 2$ m, where the letter m is used to represent both mass and the units of length, metres. In all material for this course, and in most other printed text, letters used to represent physical quantities are printed in italics, whereas those used for units are not.

3.5.3 Reading equations

To understand, and thus use, the equations in Box 3.4 you need to be aware of a few rules and conventions. Most of these are extensions of things you have learnt earlier in this course. First:

> When using symbols instead of words or numbers, it is conventional to drop the '$\times$' sign for multiplication.

So in Equation 3.6, ma means mass *times* magnitude of acceleration and in Equation 3.11, ρgh means density *times* acceleration due to gravity *times* depth.

> Rules of arithmetic, such as the fact that addition and multiplication are commutative, and the BEDMAS order of operations, apply when using symbols too.

The fact that multiplication is commutative means that equations involving several multiplications can be written in any order. So Equation 3.14 could be (and sometimes is) written as $q = cm\,\Delta T$ instead of $q = mc\,\Delta T$. Addition is also commutative, so Equation 3.16 could be written as $v_x = a_x t + u_x$ instead of $v_x = u_x + a_x t$.

Although the order in which multiplications are written doesn't matter, various conventions are generally applied. Note that in Equation 3.3 ($C = 2\pi r$), the number 2 is written first, then the constant π, then the variable r. This order (numbers, then constants, then variables) is the one that is generally applied. Similarly, $E = mc^2$ (Equation 3.7) could be written as $E = c^2 m$, but it generally isn't! Variables that are raised to a power tend to appear at the end of equations.

BEDMAS tells us that operations within brackets take precedence, i.e. operations inside brackets should be evaluated before those outside the brackets. When working with symbols, this means that an operation applied to a bracket applies to everything within the bracket. So in Equation 3.19, the whole of $\left(\frac{2GM}{R}\right)$ is raised to the power $\frac{1}{2}$. Equation 3.20 uses two sets of brackets (different styles of brackets have been used to avoid confusion). The inner, round brackets ( ) are used to indicate that L should be divided by the whole of $(4\pi F)$ and the outer, square brackets [ ] are used to indicate that the whole of $L/(4\pi F)$ should be raised to the power $\frac{1}{2}$.

There are two further points to note that are linked to the use of brackets.

1 A square root sign and a horizontal line used to indicate division can both be thought of as containing invisible brackets, i.e. the square root sign is taken to apply to everything within the sign and the division applies to everything above the line. So, in Equation 3.10, the square root applies to the whole of $\left(\dfrac{\mu}{\rho}\right)$, (this means that $\sqrt{\dfrac{\mu}{\rho}}$ could be written as $\dfrac{\sqrt{\mu}}{\sqrt{\rho}}$), and in Equation 3.15 the whole of $(v_\mathrm{i} + v_\mathrm{f})$ should be divided by two.

2 Throughout this course, brackets are sometimes used for added clarity even when this is not strictly necessary. In addition, you are encouraged to add your own brackets whenever you think doing so would make the meaning of an equation clearer.

The 'E' in BEDMAS (see Section 1.4) tells us that exponents take precedence over divisions and multiplications, so in Equation 3.7 ($E = mc^2$) the c must be squared before being multiplied by m. This means that it is *only* the c that is squared, not the m. For clarity you could write this as $E = m\left(c^2\right)$, but it is very important to remember that $mc^2 \neq (mc)^2$, i.e. that $mc^2 \neq m^2c^2$, where the symbol $\neq$ means 'is *not* equal to'.

BEDMAS also reminds us that multiplications should be carried out before additions and subtractions, so in Equation 3.16, a_x and t should be multiplied together before u_x is added.

Finally, note that all of the rules discussed in Chapter 1 for the writing and manipulation of fractions and powers apply when using symbols, in exactly the same way as they do when using numbers. So, Equation 3.17 could be written as $s_x = u_xt + \dfrac{a_xt^2}{2}$ instead of $s_x = u_xt + \frac{1}{2}a_xt^2$; Equation 3.18 could be written as $F_{\rm g} = \dfrac{Gm_1m_2}{r^2}$ instead of $F_{\rm g} = G\,\dfrac{m_1m_2}{r^2}$; and the following two representations of Equation 3.20, although they look very different, are actually identical in meaning:

$$d = \sqrt{\frac{L}{4\pi F}} \qquad\qquad d = [L/(4\pi F)]^{1/2}$$

Question 3.11 Answer

Which *two pairs* of equations for a of those given below are equivalent? You should be able to answer this question by just looking at the equations, but you might like to check your answer by substituting values such as $x = 3$, $y = 4$, $z = 5$.

(i) $a = x(y + z)$

(ii) $a = xy + z$

(iii) $a = (y + z)x$

(iv) $a = x + yz$

(v) $a = z + yx$

Question 3.12 Answer

Two of the equations given below for m are equivalent. Which two? Again, you should attempt this question initially by simply looking at the equations.

(i) $m = \dfrac{bac^2}{d}$

(ii) $m = a\dfrac{b^2c^2}{d}$

(iii) $m = a\dfrac{bc^2}{d}$

(iv) $m = \dfrac{abc^2}{ad}$

(v) $m = \dfrac{b^2a^2c^2}{d}$

3.5.4 Using equations

Substituting values into equations provides a way of checking your understanding of many of the techniques introduced in this chapter, especially the correct reading of equations, the use of scientific notation, and the need to quote answers to an appropriate number of significant figures. It also provides an opportunity for you to extend your understanding of units in calculations and to begin to think about how to choose an appropriate equation to use in answering a particular question. Don't worry about the science in the worked examples in this section; they are given as illustrations of good practice for substituting values into equations.

Worked example 3.8

Use $v_x = u_x + a_x t$ (Equation 3.16) to find the speed reached after 0.45 s by a stone thrown downwards from a cliff with initial speed $1.5\ \mathrm{m\,s^{-1}}$. This situation is illustrated in Figure 3.11. You can assume that the magnitude (size) of the acceleration is $9.81\ \mathrm{m\,s^{-2}}$, where $\mathrm{m\,s^{-2}}$ are the SI units of acceleration.

Answer

Equation 3.16 states that $v_x = u_x + a_x t$, and we are trying to find v_x. The question tells us that

$$u_x = 1.5\ \mathrm{m\,s^{-1}} \quad a_x = 9.81\ \mathrm{m\,s^{-2}} \quad t = 0.45\ \mathrm{s}$$

Thus

$$v_x = \left(1.5\ \mathrm{m\,s^{-1}}\right) + \left(9.81\ \mathrm{m\,s^{-2}} \times 0.45\ \mathrm{s}\right)$$

where the units of a_x are $\mathrm{m\,s^{-2}}$ and the units of t are s, so the units of $a_x t$ are $\mathrm{m\,s^{-2}} \times \mathrm{s}$. Simplifying this gives

$$\mathrm{m\,s^{-2}} \times \mathrm{s} = \frac{\mathrm{m}}{\mathrm{s^2}} \times \mathrm{s} = \frac{\mathrm{m} \times \cancel{\mathrm{s}}}{\mathrm{s} \times \cancel{\mathrm{s}}} = \frac{\mathrm{m}}{\mathrm{s}} = \mathrm{m\,s^{-1}}$$

So

$$\begin{aligned} v_x &= 1.5\ \mathrm{m\,s^{-1}} + 4.4145\ \mathrm{m\,s^{-1}} \\ &= 5.9\ \mathrm{m\,s^{-1}} \text{ to two significant figures,} \end{aligned}$$

i.e. the speed after 0.45 seconds is $5.9\ \mathrm{m\,s^{-1}}$.

Note, from Worked example 3.8, the following points about the handling of units:

1 Calculations have been done in SI units.

2 Units have been included next to values at all times, and the units in the final answers are both consistent with the working *and* what we would expect the units of the final answer to be.

The second point follows from what was said about units in Section 3.1.1; we have input values with units of $\mathrm{m\,s^{-1}}$ for initial speed, units of s for time, and units of $\mathrm{m\,s^{-2}}$ for acceleration, and the units for final speed have *worked out to be* $\mathrm{m\,s^{-1}}$. We have not simply assumed the units for final speed to be $\mathrm{m\,s^{-1}}$, but rather have calculated the units for v_x at the same time as calculating the numerical value. Handling units in this way ensures that the answers are expressed as physical quantities (with units), not just numbers. It also gives an easy way of checking a calculation. If the final units in Worked example 3.8 had come out as $\mathrm{m^2\,s^{-1}}$ you might have realized that, since these are *not* units of speed, you must have made a mistake.

It is good practice to work out the units in this way in *all* your scientific calculations. To enable you to do this, Box 3.5 explains a little more about some of the derived units that you will encounter in this course.

Box 3.5 Derived SI units

Box 2.1 introduced the SI base units, and since then you have encountered the SI units of $m\,s^{-1}$ for speed, $kg\,m^{-3}$ for density and $m\,s^{-2}$ for acceleration. These units are combinations of the base units m, kg and s; other physical quantities have units involving other base units too. Some physical quantities are so commonly used that their units have names and symbols of their own, even though they could be stated as a combination of base units. Several of these derived units are listed in Table 3.2. Note that if you become a sufficiently famous scientist you are likely to end up with a unit named after you! The units in Table 3.2 are named after Sir Isaac Newton, James Prescott Joule, James Watt, Blaise Pascal and Heinrich Hertz respectively.

Physical quantity	Name of unit	Symbol for unit	Base unit equivalent
force, such as weight	newton	N	$kg\,m\,s^{-2}$
energy	joule	J	$kg\,m^2\,s^{-2}$
power	watt	W	$kg\,m^2\,s^{-3}$
pressure	pascal	Pa	$kg\,m^{-1}\,s^{-2}$
frequency	hertz	Hz	s^{-1}

Table 3.2: Some derived units

Note also that many of the derived units are interlinked:

$$1\ \text{J} = 1\ \text{N} \times 1\ \text{m}$$

$$1\ \text{W} = \frac{1\ \text{J}}{1\ \text{s}}$$

$$1\ \text{Pa} = \frac{1\ \text{N}}{1\ \text{m}^2}$$

The following data may help to illustrate the sizes of the units:

- An eating apple has a weight of about 1 N on Earth;
- An athlete with mass 75 kg, sprinting at 9 $\text{m}\,\text{s}^{-1}$, has an energy of about 3000 J;
- A domestic kettle has a power rating of about 2500 W;
- Atmospheric pressure at sea-level is about 10^5 Pa;
- The human heart beats with a frequency of about 1.3 Hz.

To find the units of v_{esc} in Worked example 3.9, you need to use the fact, from Table 3.2, that $1\ \mathrm{N} = \mathrm{kg\,m\,s^{-2}}$. This worked example also provides a reminder of the importance of converting to SI base units before beginning a calculation.

Worked example 3.9

Use $v_{esc} = \left(\dfrac{2GM}{R}\right)^{1/2}$ (Equation 3.19) to find the escape speed, v_{esc}, needed for an object to escape from the Earth's gravitational attraction. The mass of the Earth, $M = 5.98 \times 10^{24}$ kg, the radius of the Earth, $R = 6.38 \times 10^{3}$ km and $G = 6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$.

Answer

Converting R to SI base units gives

$$\begin{aligned} R &= 6.38 \times 10^{3}\ \mathrm{km} \\ &= 6.38 \times 10^{3} \times 10^{3}\ \mathrm{m} \\ &= 6.38 \times 10^{6}\ \mathrm{m} \end{aligned}$$

$$M = 5.98 \times 10^{24}\ \mathrm{kg}$$

$$G = 6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$$

Substituting in Equation 3.19

$$v_{\mathrm{esc}} = \left(\frac{2GM}{R}\right)^{1/2}$$
$$= \left(\frac{2 \times 6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}} \times 5.98 \times 10^{24}\ \mathrm{kg}}{6.38 \times 10^{6}\ \mathrm{m}}\right)^{1/2}$$

Rearranging this so that the units on the top of the fraction are all together we get

$$v_{\mathrm{esc}} = \left(\frac{2 \times 6.673 \times 10^{-11} \times 5.98 \times 10^{24}\,\mathrm{N\,m^2\,kg^{-2}\,kg}}{6.38 \times 10^{6}\ \mathrm{m}}\right)^{1/2}$$

Since 1 N = 1 $\mathrm{kg\,m\,s^{-2}}$, this can be rewritten as

$$v_{\mathrm{esc}} = \left(\frac{2 \times 6.673 \times 10^{-11} \times 5.98 \times 10^{24}\,\mathrm{kg\,m\,s^{-2}\,m^2\,kg^{-2}\,kg}}{6.38 \times 10^{6}\ \mathrm{m}}\right)^{1/2}$$

This can be simplified by cancelling some of the units

$$v_{\mathrm{esc}} = \left(\frac{2 \times 6.673 \times 10^{-11} \times 5.98 \times 10^{24}\,\cancel{\mathrm{kg}}\,\cancel{\mathrm{m}}\,\mathrm{s^{-2}\,m^2}\,\cancel{\mathrm{kg^{-2}}}\,\cancel{\mathrm{kg}}}{6.38 \times 10^{6}\ \cancel{\mathrm{m}}}\right)^{1/2}$$

Calculating the numeric value, and reordering the units, we have

$$v_{\text{esc}} = \left(1.2509 \times 10^8\ \text{m}^2\ \text{s}^{-2}\right)^{1/2}$$

Taking the square root of both 1.2509×10^8 and $\text{m}^2\ \text{s}^{-2}$ gives

$$v_{\text{esc}} = 1.12 \times 10^4\ \text{m s}^{-1} \text{ to three significant figures.}$$

The escape speed is $1.12 \times 10^4\ \text{m s}^{-1}$, with units of m s^{-1}, as expected.

Question 3.13 Answer

In a classic experiment in the USA in 1926, Edgar Transeau calculated the amount of energy stored in the corn plants in a one-acre field in a 100-day growing period to be 1.06×10^8 kJ. This is *NPP* in Equation 3.8. For the same field and the same time period, he found the energy used by the plants in respiration (R) to be 3.23×10^7 kJ. Use Equation 3.8 to find the corresponding value of *GPP*, the total energy captured by the plants.

Question 3.14 Answer

Use Equation 3.13 to find the speed of waves (in $\mathrm{m\,s^{-1}}$) which have a frequency of 4.83×10^{14} Hz and a wavelength of 621 nm.

The final worked example in this section returns us to the piece of granite introduced at the beginning of the chapter. It is perhaps a somewhat more realistic example than Worked examples 3.8 and 3.9 because the question does not tell us which formula to use.

Worked example 3.10

The rigidity modulus of granite (a measure of the rock's ability to resist deformation) near the surface of the Earth is 3.0×10^{10} Nm^{-2}. Use this value, and the value you found previously for the density of granite to find the speed of S waves travelling through granite.

Answer

Which equation shall we use? When faced by this dilemma it is best to start by thinking carefully about what you already know and what you want to find. On this occasion we're told that the rigidity modulus is 3.0×10^{10} Nm^{-2} and we know (from Worked example 3.7) that the density of granite is 2.7×10^{3} kg m^{-3} (using a value to three significant figures to avoid rounding errors). We need to find a value for S wave speed. So we need an equation which links density, rigidity modulus and S wave speed; Equation 3.10 ($v_{\text{S}} = \sqrt{\frac{\mu}{\rho}}$) from Box 3.4 fits the bill.

Simply finding an equation from a list, all that is possible in this course, is somewhat unlike the situation you are likely to encounter in the real scientific world. Nevertheless, the principle of starting each question by thinking about what you already know and what you want to find is a good one, and on this occasion it makes it straightforward to find an equation to use from Box 3.4.

$$v_s = \sqrt{\frac{\mu}{\rho}}$$

$$\mu = 3.0 \times 10^{10}\ \mathrm{N\,m^{-2}}$$

$$\rho = 2.70 \times 10^{3}\ \mathrm{kg\,m^{-3}}$$

So

$$v_s = \sqrt{\frac{3.0 \times 10^{10}\ \mathrm{N\,m^{-2}}}{2.70 \times 10^{3}\ \mathrm{kg\,m^{-3}}}}$$

Since 1 N = 1 $\mathrm{kg\,m\,s^{-2}}$, this can be rewritten as

$$v_s = \sqrt{\frac{3.0 \times 10^{10}\ \mathrm{kg\,m\,s^{-2}\,m^{-2}}}{2.70 \times 10^{3}\ \mathrm{kg\,m^{-3}}}}$$

This can be simplified by cancelling the kg on top and bottom of the fraction

$$v_s = \sqrt{\frac{3.0 \times 10^{10}\ \cancel{\mathrm{kg}}\,\mathrm{m\,s^{-2}\,m^{-2}}}{2.70 \times 10^{3}\ \cancel{\mathrm{kg}}\,\mathrm{m^{-3}}}}$$

Calculating the numeric value, and combining the m and m^{-2} on the top of the fraction with the m^{-3} on the bottom, we have

$$\begin{aligned} v_s &= \sqrt{1.11 \times 10^7\ \mathrm{m^2\,s^{-2}}} \\ &= 3.3 \times 10^3\ \mathrm{m\,s^{-1}} \text{ to two significant figures} \end{aligned}$$

So the S waves travel with a speed of $3.3 \times 10^3\ \mathrm{m\,s^{-1}}$ through granite.

Question 3.15

The Earth has an average radius of 6.38×10^3 km and a mass of 5.97×10^{24} kg. The Moon has a mass of 7.35×10^{22} kg. The distance between the Earth and the Moon is 3.84×10^5 km and $G = 6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$. Use appropriate equations from Box 3.4 to calculate:

(a) the Earth's volume (in m^3); Answer

(b) the magnitude of the gravitational force between the Earth and the Moon (in newtons). Answer

Note: on this occasion you should be able to work out the final units of your answer without expressing newtons in the form of base units. This is further discussed in the answer to the question.

3.6 Learning outcomes for Chapter 3

After completing your work on this chapter you should be able to:

3.1 demonstrate understanding of the terms emboldened in the text;

3.2 perform calculations to an appropriate number of significant figures;

3.3 give answers to calculations in appropriate SI units;

3.4 carry out calculations in scientific notation, both with and without the use of a scientific calculator;

3.5 estimate answers to one significant figure;

3.6 convert between various units for quantities such as area, volume, speed, density and concentration;

3.7 demonstrate understanding of the rules and conventions used in scientific formulae;

3.8 substitute values (numbers and units) into scientific formulae.

4

Algebra

At the end of Chapter 3 we used the equation $v_s = \sqrt{\frac{\mu}{\rho}}$ to calculate the S wave speed, v_s, of seismic waves passing through a rock of density ρ and rigidity modulus μ. But suppose that, instead of knowing ρ and μ and wanting to find v_s, we know v_s and ρ and want to find μ. The best way to proceed is to rearrange $v_s = \sqrt{\frac{\mu}{\rho}}$ to make μ the subject of the equation, where the word 'subject' is used to mean the term written by itself, usually to the left of the equals sign. Rearranging equations is the first topic considered in Chapter 4. The rest of the chapter introduces methods for simplifying equations and ways of combining two or more equations together, and it ends with a look at ways of using algebra to solve problems.

4.1 Rearranging equations

There are many different methods taught for rearranging equations, and if you are happy with a method you have learnt previously it is probably best to stick with this method, provided it gives correct answers to all the questions in this section. However, if you have not found a way of rearranging equations which suits you, you might like to try the method highlighted in the blue-toned boxes throughout this section. This method draws on an analogy between an equation and an old-fashioned set of kitchen scales, and considers the equation to be 'balanced' at the equals sign. The scales will remain balanced if you add a 50 g mass to one side of the scales, or halve the mass on one side, *provided* you do exactly the same thing to the other side. In a similar way, you can do (almost) anything you like to one side of an equation and, provided you do exactly the same thing to the other side, the equation will still be valid. This point is illustrated in Figure 4.1.

The following rule summarizes the discussion above:

> Whatever you do mathematically to one side of an equation you must also do to the other side.

This rule is fundamental when rearranging equations, but it doesn't tell you *what* operation to perform to both sides of an equation in order to rearrange it in the way you want. The highlighted points below should help with this, as will plenty of practice.

Two things are worth noting at the outset:

1 Equations are conventionally written with the subject on the left-hand side of the equals sign. However, when rearranging an equation it can very often be helpful simply to reverse the order.

So if you derive or are given the equation $c = a+b$ you can rewrite it as $a+b = c$; if you derive or are given the equation $ab = c$ you can rewrite it as $c = ab$.

2 Even if you choose the 'wrong' operation, provided you correctly perform that operation to both sides of the equation, the equation will still be valid. Suppose we want to rearrange the equation $c = a + b$ to obtain an expression for a. We could divide by two, as illustrated by Figure 4.1c; this gives

$$\frac{c}{2} = \frac{a+b}{2}$$

This is a perfectly valid equation; it just doesn't help much in our quest for a. The numbered points below give some hints for more helpful ways forward, and each guideline is followed by an illustration of its use.

In the numbered hints the words expression and term are used to describe the parts of an equation. An equation must always include an equals sign, but an expression or term won't. A term may be a single variable (such as v_x or u_x in the equation $v_x = u_x + a_x t$, or a combination of several variables (such as $a_x t$); an expression is usually a combination of variables (such as $a_x t$ or $u_x + a_x t$, but the words are often used interchangeably.

Hint 1

If you want to remove an expression that is *added* to the term you want, *subtract* that expression from both sides of the equation.

To rearrange $a + b = c$ to make a the subject, note that we need to remove the b from the left-hand side of the equation. The b is currently added to a, so we need to subtract b from both sides. This gives

$$a + b - b = c - b$$

or

$$a = c - b \quad (\text{since } b - b = 0)$$

Hint 2

If you want to remove an expression that is *subtracted* from the term you want, *add* that expression to both sides of the equation.

To rearrange $a - b = c$ to make a the subject, note that we need to remove the b from the left-hand side of the equation. The b is currently subtracted from a, so we need to add b to both sides. This gives

$$a - b + b = c + b$$

or

$$a = c + b \quad (\text{since } -b + b = 0)$$

Hint 3

If the term you want is *multiplied* by another expression, *divide* both sides of the equation by that expression.

To rearrange $ab = c$ to make a the subject, note that we need to remove the b from the left-hand side of the equation. The a is currently multiplied by b, so we need to divide both sides of the equation by b. This gives

$$\frac{ab}{b} = \frac{c}{b}$$

The b in the numerator of the fraction on the left-hand side cancels with the b in the denominator to give

$$a = \frac{c}{b}$$

Hint 4

If the term you want is *divided* by another expression, *multiply* both sides of the equation by that expression.

To rearrange $\frac{a}{b} = c$ to make a the subject, note that we need to remove the b from the left-hand side of the equation. The a is currently divided by b, so we need to multiply both sides of the equation by b. This gives

$$\frac{a \times b}{b} = c \times b$$

The b in the numerator of the fraction on the left-hand side cancels with the b in the denominator to give

$$a = cb$$

Hint 5

If you are trying to make a term the subject of an equation and you currently have an equation for the *square* of that term, take the *square root* of both sides of the equation.

To rearrange $a^2 = b$ to make a the subject, note that the a is currently squared, and take the square root of both sides of the equation. This gives

$$a = \pm\sqrt{b}$$

Note the presence of the $\pm$ sign, indicating that the answer could be either positive or negative, as discussed in Section 1.1.3. In practice, the reality of the problem we are solving sometimes allows us to rule out one of the two values.

Hint 6

If you are trying to make a term the subject of an equation and you currently have an equation for the *square root* of that term, *square* both sides of the equation.

To rearrange $\sqrt{a} = b$ to make a the subject, note that you currently have an equation for the square root of a, and square both sides of the equation. This gives

$$a = b^2$$

Hints 1 to 6 all follow from a general principle:

To 'undo' an operation (e.g. $+$, $-$, $\times$, $\div$, square, square root) you should do the opposite, (i.e. $-$, $+$, $\div$, $\times$, square root, square).

The following worked examples use the principles introduced in the numbered hints above, in the context of equations which are frequently encountered in science. Worked example 4.1 also involves substituting numerical values and units into the equation once it has been rearranged.

Worked example 4.1

As discussed in Box 2.1, mass and weight are not the same. However, the magnitude of the weight, W, of an object at the surface of the Earth and its mass, m, are related by the equation $W = mg$. The magnitude of the acceleration due to gravity, g, can be taken as $9.81\ \mathrm{m\,s^{-2}}$

A teenager's weight is 649 N. What is his mass?

Answer

We need to start by rearranging $W = mg$ to make m the subject of the equation. It is helpful to start by reversing the order of the equation, i.e. to write it as

$$mg = W$$

To isolate m we need to get rid of g, and m is currently *multiplied* by g so, from Hint 3 we need to *divide* by g. Remember that we must do this to *both sides of the equation*, so we have

$$\frac{mg}{g} = \frac{W}{g}$$

The g in the numerator of the fraction on the left-hand side cancels with the g in the denominator to give

$$m = \frac{W}{g}$$

Substituting values for W and g gives

$$m = \frac{649\ \mathrm{N}}{9.81\ \mathrm{m\,s^{-2}}}$$

Since $1\ \mathrm{N} = 1\ \mathrm{kg\,m\,s^{-2}}$ (from Table 3.2) and

$$\frac{\mathrm{N}}{\mathrm{m\,s^{-2}}} = \frac{\mathrm{kg}\,\cancel{\mathrm{m}}\,\cancel{\mathrm{s^{-2}}}}{\cancel{\mathrm{m}}\,\cancel{\mathrm{s^{-2}}}}$$

we then have

$$m = \frac{649\ \mathrm{kg\,m\,s^{-2}}}{9.81\ \mathrm{m\,s^{-2}}} = 66.2\ \mathrm{kg}$$

So the teenager’s mass is 66.2 kg

Worked example 4.2

The time T for one swing of a pendulum is related to its length, L, by the equation

$$T^2 = \frac{4\pi^2 L}{g}$$

where g is the magnitude of the acceleration due to gravity. Write down an equation for T.

Answer

T is currently squared, so from Hint 5, we need to take the square root of both sides of the equation. This gives

$$T = \sqrt{\frac{4\pi^2 L}{g}}$$

Since T is a period of time, its value must be positive, so we only need to write down the positive square root.

Question 4.1

(a) Rearrange $v = f\lambda$ to make f the subject. Answer

(b) Rearrange $E_{\text{tot}} = E_{\text{k}} + E_{\text{p}}$ so that E_{k} is the subject. Answer

(c) Rearrange $\rho = \dfrac{m}{V}$ to obtain an equation for m. Answer

When rearranging more complicated equations, it is often necessary to proceed in several steps. Each step will use the rules already discussed, but many people are perplexed when trying to decide which step to take first. Expertise in this area comes largely with practice, and there are no hard and fast rules (it is often possible to rearrange an equation by several, equally correct, routes). However, the following guidelines may help:

Hint 7

Don't be afraid of using several small steps to rearrange one equation.

Hint 8

Aim to get the new subject into position on the left-hand side as soon as you can. (This will not always be possible straight away.) Simply reversing an equation can sometimes be a helpful initial step.

Hint 9

You can treat an expression within brackets as if it was a single term. This is true whether the brackets are shown explicitly in the original equation or whether you have added them (or imagined them) for clarity. If the quantity required as the subject is itself part of an expression in brackets in the original equation, it is often best to start by making the whole bracketed term the subject of the equation.

Let's look at these guidelines in the context of a series of worked examples, interspersed with questions for you to try for yourself. Note that although 'real' science equations have been used as much as possible in the worked examples and questions, the symbols have not been explained, and you do not need to know the meaning of them. This is to allow you to concentrate, *for the time being only*, on the algebra rather than getting side-tracked into the underlying science.

You may be able to rearrange the equations in the following worked examples in fewer steps than are shown, but if you are in any doubt at all it is best to write down all the intermediate steps in the process.

Worked example 4.3

Rearrange $PV = nRT$ to give an equation for T.

Answer

This example is perhaps more straightforward than it looks, but it is best to proceed in steps.

The first step is to reverse the equation so that the T is on the left-hand side (from Hint 8). This gives

$$nRT = PV$$

We now need to remove the nR by which the T is multiplied. Dividing both sides by nR gives

$$\frac{nRT}{nR} = \frac{PV}{nR}$$

The nR in the numerator of the fraction on the left-hand side cancels with the nR in the denominator to give

$$T = \frac{PV}{nR}$$

Worked example 4.4

Rearrange $\rho = \dfrac{m}{V}$ so that V is the subject.

Answer

The first step is to multiply both sides by V (thus getting V into the right position, as in Hint 8). This gives

$$\rho V = \frac{mV}{V}$$

that is

$$\rho V = m$$

Then dividing both sides by ρ gives

$$\frac{\rho V}{\rho} = \frac{m}{\rho}$$

that is

$$V = \frac{m}{\rho}$$

Worked example 4.5

Rearrange $v_x = u_x + a_x t$ to make u_x the subject.

Answer

This equation can be written as

$$u_x + a_x t = v_x$$

which has u_x on the left-hand side (Hint 8).

We can treat the expression $a_x t$ as a single term (by considering there to be brackets around it, as in Hint 9) and subtract it from both sides of the equation to give

$$u_x + a_x t - a_x t = v_x - a_x t$$

that is

$$u_x = v_x - a_x t$$

Worked example 4.6

Rearrange $h = \frac{1}{2}gt^2$ to give an equation for t.

Answer

We can consider there to be brackets around (t^2) and start by finding an expression for t^2 (Hint 9). The equation can be written as

$$\frac{1}{2}gt^2 = h$$

which has t^2 on the left-hand side (Hint 8). Multiplying both sides by 2 gives

$$2 \times \frac{1}{2}gt^2 = 2h$$

that is

$$gt^2 = 2h$$

Dividing both sides by g gives

$$\frac{gt^2}{g} = \frac{2h}{g}$$

that is

$$t^2 = \frac{2h}{g}$$

Now we can take the square root of both sides to give

$$t = \pm\sqrt{\frac{2h}{g}}$$

Worked example 4.7

Rearrange $v_{\mathrm{s}} = \sqrt{\dfrac{\mu}{\rho}}$ so that μ is the subject.

Answer

We can consider there to be brackets around $\left(\dfrac{\mu}{\rho}\right)$ and start by finding an expression for $\left(\dfrac{\mu}{\rho}\right)$ (Hint 9).

The equation can be written as $\sqrt{\dfrac{\mu}{\rho}} = v_{\mathrm{s}}$, which has $\dfrac{\mu}{\rho}$ on the left-hand side (Hint 8).

Squaring both sides gives

$$\frac{\mu}{\rho} = v_{\mathrm{s}}^2$$

Now we can multiply both sides by ρ, to give $\mu = v_{\mathrm{s}}^2\,\rho$.

Box 4.1 Interlude: why bother with algebra?

You may have recognized the equation rearranged in Worked example 4.7; it was the one discussed at the beginning of the chapter. Thinking back to the beginning of the chapter reminds us of the purpose of what we are doing. The ability to rearrange equations is useful (arguably the most useful skill developed in this course), but it's not something that you should do just for the sake of doing so, but rather because you want to work something out, and rearranging an equation is the means to this end. Suppose you have been told that S waves pass through rocks of density $\rho = 3.9 \times 10^3\ \mathrm{kg\,m^{-3}}$ with a speed $v_\mathrm{s} = 3.0 \times 10^3\ \mathrm{m\,s^{-1}}$, and you want to find the rigidity modulus μ. The equation in the form $v_\mathrm{s} = \sqrt{\dfrac{\mu}{\rho}}$ is not much use, but the rearranged form immediately tells us that

$$\begin{aligned}\mu &= v_\mathrm{s}^2\,\rho \\ &= \left(3.0 \times 10^3\ \mathrm{m\,s^{-1}}\right)^2 \times \left(3.9 \times 10^3\ \mathrm{kg\,m^{-3}}\right) \\ &= 3.5 \times 10^{10}\ \mathrm{m^2\,s^{-2}\,kg\,m^{-3}} \\ &= 3.5 \times 10^{10}\ \mathrm{kg\,m^{-1}\,s^{-2}}\end{aligned}$$

So the rigidity modulus is $3.5 \times 10^{10}\ \mathrm{kg\,m^{-1}\,s^{-2}}$.

Question 4.2

(a) Rearrange $b = c - d + e$ so that e is the subject. Answer

(b) Rearrange $p = \rho g h$ to give an equation for h. Answer

(c) Rearrange $v_{\text{esc}}^2 = \dfrac{2GM}{R}$ to make R the subject. Answer

(d) Rearrange $E = hf - \phi$ so that ϕ is the subject. Answer

(e) Rearrange $a = \dfrac{bc^2}{d}$ to give an equation for c. Answer

(f) Rearrange $a = \sqrt{\dfrac{b}{c}}$ to make b the subject. Answer

Question 4.3

The mass, m, speed, v, and kinetic energy, E_k, of an object are linked by the equation $E_\mathrm{k} = \frac{1}{2}mv^2$.

(a) Rearrange this equation so that v is the subject. Answer

(b) Use your answer to part (a) to estimate (in $\mathrm{m\,s^{-1}}$ to one significant figure) the speed needed in order for a tectonic plate of mass 4×10^{21} kg to have a kinetic energy of 2×10^3 J. Answer

(c) Use your answer to part (a) to estimate (in $\mathrm{m\,s^{-1}}$ to one significant figure) the speed needed in order for an athlete of mass 70 kg to have the same kinetic energy as the tectonic plate in part (b). Answer

The final group of worked examples in this section involve equations which may appear rather more complex than the previous ones, but they can all be rearranged using the rules and guidelines already introduced. Some, like Worked example 4.8, appear more complex partly because they use symbols that are rather unwieldy. However, these final worked examples are genuinely more complicated too, and are best solved by taking a logical stepwise approach (as the early Arab mathematicians did; see Box 4.2). Rearranging complicated equations is rather like peeling away layers of an onion, systematically removing layer by layer in order to get to the part you want. But that doesn't mean it should end in tears!

Box 4.2 Al-Khwarizmi and al-jabr

The techniques of algebra have developed over a period of several thousand years, but the word 'algebra' comes from 'al-jabr' in the title of a book written by Mohammed ibn-Musa al-Khwarizmi in about 830. The book, whose title *Hisab al-jabr w'al muqabela*, can be translated as 'Transposition and reduction', explained how it was possible to reduce any problem to one of six standard forms using the two processes, al-jabr (transferring terms to eliminate negative quantities) and muqabela (balancing the remaining positive quantities).

Arab mathematicians like al-Khwarizmi did not use symbols in their work, but rather explained everything in words. Nevertheless, their stepwise approach was very similar to the one advocated in this course. Al-Khwarizmi is also remembered for his work on the solution of quadratic equations, discussed later in this chapter.

A little less working is shown in Worked examples 4.8, 4.9 and 4.10 than previously, and hints are not explicitly referred to. This has been done so as to make the working more akin to what you might reasonably write when working through the questions in this course. You are encouraged to show as many steps as necessary in your working, and to use words of explanation wherever they help you.

Worked example 4.8

Rearrange $\Delta G_\mathrm{m}^\ominus = \Delta H_\mathrm{m}^\ominus - T\Delta S_\mathrm{m}^\ominus$ so that $\Delta S_\mathrm{m}^\ominus$ is the subject.

(*Note:* $\Delta G_\mathrm{m}^\ominus$, $\Delta H_\mathrm{m}^\ominus$ and $\Delta S_\mathrm{m}^\ominus$ each represent a single physical entity.)

Answer

Adding $T\Delta S_\mathrm{m}^\ominus$ to both sides of the equation gives

$$\Delta G_\mathrm{m}^\ominus + T\Delta S_\mathrm{m}^\ominus = \Delta H_\mathrm{m}^\ominus$$

Subtracting $\Delta G_\mathrm{m}^\ominus$ from both sides gives

$$T\Delta S_\mathrm{m}^\ominus = \Delta H_\mathrm{m}^\ominus - \Delta G_\mathrm{m}^\ominus$$

Dividing both sides by T gives

$$\Delta S_\mathrm{m}^\ominus = \frac{\Delta H_\mathrm{m}^\ominus - \Delta G_\mathrm{m}^\ominus}{T}$$

Worked example 4.9

Rearrange $s_x = u_x t + \frac{1}{2} a_x t^2$ to make a_x the subject.

Answer

The equation can be written as $u_x t + \frac{1}{2} a_x t^2 = s_x$.

Subtracting $u_x t$ from both sides gives

$$\frac{1}{2} a_x t^2 = s_x - u_x t$$

Multiplying both sides by 2 gives

$$a_x t^2 = 2(s_x - u_x t)$$

Dividing both sides by t^2 gives

$$a_x = \frac{2(s_x - u_x t)}{t^2}$$

Worked example 4.10

Rearrange $F_{\mathrm{g}} = G\dfrac{m_1 m_2}{r^2}$ to give an equation for r.

Answer

Note that $F_{\mathrm{g}} = G\dfrac{m_1 m_2}{r^2}$ can be written as $F_{\mathrm{g}} = \dfrac{G m_1 m_2}{r^2}$ (see Section 3.5.3).

We can get the r^2 into position on the left-hand side by multiplying both sides by r^2. This gives

$$F_{\mathrm{g}} r^2 = G m_1 m_2$$

Dividing both sides by F_{g} gives

$$r^2 = \frac{G m_1 m_2}{F_{\mathrm{g}}}$$

Taking the square root of both sides gives

$$r = \pm\sqrt{\frac{G m_1 m_2}{F_{\mathrm{g}}}}$$

Box 4.3 Using algebra in astronomy

The luminosity of a star (the total rate at which it radiates energy into space, in all directions), L, is related to its radius, R, and the temperature (measured in kelvin), T, of its outer layer (called the photosphere) by the equation

$$L = 4\pi R^2 \sigma T^4 \tag{4.1}$$

where σ (the lower case Greek letter sigma) represents a constant known as Stefan's constant, with the value $\sigma = 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$.

It is impossible to take direct readings for the luminosity, radius or temperature of distant stars, but indirect measurements can lead to values for photospheric temperature and luminosity. Figure 4.2 is a so-called Hertzsprung–Russell diagram, comparing the photospheric temperatures and luminosity of different stars. Note that different types of stars appear in distinct groupings on the Hertzsprung–Russell diagram.

If we know a star's luminosity and photospheric temperature we can find its radius from Equation 4.1, but first of all we need to rearrange the equation to make R the subject.

Equation 4.1 can be reversed to give

$$4\pi R^2 \sigma T^4 = L$$

Dividing both sides by $4\pi\sigma T^4$ gives

$$R^2 = \frac{L}{4\sigma T^4}$$

(Note that the same results would have been achieved by dividing by 4, π, σ and T^4 separately.)

Taking the square root of both sides gives

$$R = \pm\sqrt{\frac{L}{4\sigma T^4}}$$

Since R is the radius of a star, we are only interested in the positive value.

The star Alcyone (in the Pleiades) has a photospheric temperature of 1.2×10^4 K and a luminosity of 3.2×10^{29} W. So its radius is

$$\begin{aligned} R &= \sqrt{\frac{3.2 \times 10^{29}\ \mathrm{W}}{4 \times 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}} \times (1.2 \times 10^4\ \mathrm{K})^4}} \\ &= \sqrt{\frac{3.2 \times 10^{29}\ \cancel{\mathrm{W}}}{4 \times 5.67 \times 10^{-8}\ \cancel{\mathrm{W}}\,\mathrm{m^{-2}}\,\cancel{\mathrm{K^{-4}}}\ (1.2 \times 10^4)^4\ \cancel{\mathrm{K^4}}}} \\ &= \sqrt{2.17 \times 10^{19}\ \mathrm{m^2}} \\ &= 4.7 \times 10^9\ \mathrm{m} \end{aligned}$$

The radius of Alcyone is 4.7×10^9 m.

Notice that in this example, the units of watts cancelled without having to be expressed in SI base units.

Question 4.4

(a) Rearrange $v_x = u_x + a_x t$ so that a_x is the subject. Answer

(b) Rearrange $v_s = \sqrt{\dfrac{\mu}{\rho}}$ to make ρ the subject. Answer

(c) Rearrange $F = \dfrac{L}{4\pi d^2}$ to give an equation for d. Answer

4.2 Simplifying equations

Sometimes it is possible (and helpful) to write an algebraic expression in a different form from the one in which it is originally presented. Whenever possible you should aim to write equations in their simplest form, i.e. to simplify them. For example, you will see in this section that the equation $c = \dfrac{a}{4b} + \dfrac{3a}{4b}$ can be simplified to $c = \dfrac{a}{b}$; the latter form of the equation is rather more useful than the former.

In order to simplify equations it is often necessary to apply the rules for the manipulation of fractions and brackets that were introduced in Chapter 1.

4.2.1 Simplifying algebraic fractions

Algebraic fractions can be multiplied and divided in exactly the same way as numerical fractions, using the methods introduced in Section 1.2.4 and Section 1.2.5. So just as

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15} \quad \text{(multiplying numerators and denominators together)}$$

we can write

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

Similarly, just as

$$\begin{aligned}\frac{2}{3} \div \frac{5}{7} &= \frac{2}{3} \times \frac{7}{5} \quad \text{(turning the } \frac{5}{7} \text{ upside down and multiplying)}\\ &= \frac{2 \times 7}{3 \times 5}\\ &= \frac{14}{15}\end{aligned}$$

we can write

$$\begin{aligned}\frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \times \frac{d}{c} \quad \text{(turning the } \frac{c}{d} \text{ upside down and multiplying)}\\ &= \frac{a \times d}{b \times c}\\ &= \frac{ad}{bc}\end{aligned}$$

Worked example 4.11 illustrates a division in which several of the terms cancel.

Worked example 4.11

Simplify $\dfrac{2ab}{c} \div \dfrac{2}{c}$

Answer

Turning the $\dfrac{2}{c}$ upside down and multiplying gives

$$\frac{2ab}{c} \div \frac{2}{c} = \frac{2ab}{c} \times \frac{c}{2}$$

We can cancel the ‘2’s and the ‘c’s to give

$$\frac{2ab}{c} \div \frac{2}{c} = \frac{\cancel{2}ab}{\cancel{c}} \times \frac{\cancel{c}}{\cancel{2}} = ab$$

The method described in Section 1.2.2 for adding and subtracting numerical fractions can also be extended to algebraic fractions. We need to find a common denominator in a similar way, so, much as we can write

$$\frac{2}{3} + \frac{4}{5} = \frac{2 \times 5}{3 \times 5} + \frac{4 \times 3}{5 \times 3} = \frac{10}{15} + \frac{12}{15} = \frac{10 + 12}{15} = \frac{22}{15}$$

where the common denominator is the product of the denominators of the original fractions, we can also write

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{db} = \frac{ad + cb}{bd}$$

Worked example 4.12

Electrical resistors can be combined together in various ways. You don't need to know or understand the scientific details, but when three resistors of resistance R_1, R_2 and R_3 are combined in a particular way ('in parallel') the effective resistance is given by the term R_{eff} in the equation

$$\frac{1}{R_{\text{eff}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \tag{4.2}$$

Rearrange Equation 4.2 to make R_{eff} the subject.

Answer

We need to start by expressing the right-hand side of Equation 4.2 as a single fraction. The product of R_1, R_2 and R_3 will be a common denominator, so we can write

$$\begin{aligned}\frac{1}{R_{\text{eff}}} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \\ &= \frac{R_2R_3}{R_1R_2R_3} + \frac{R_1R_3}{R_1R_2R_3} + \frac{R_1R_2}{R_1R_2R_3} \\ &= \frac{R_2R_3 + R_1R_3 + R_1R_2}{R_1R_2R_3}\end{aligned}$$

In order to make R_{eff} the subject of the equation, rather than $\frac{1}{R_{\text{eff}}}$, we could multiply and divide both sides of the equations by a series of expressions. However, it is more straightforward simply to turn the equation upside down, i.e. to take the reciprocal of both sides. This gives

$$R_{\text{eff}} = \frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}$$

A note of caution when simplifying algebraic expressions

When you simplify an algebraic expression, especially one involving fractions, the answer you arrive at doesn't always look very simple! If you are asked to simplify an expression which is the sum or product of two separate fractions, your answer should normally be a *single* fraction, but an expression like

$$R_{\text{eff}} = \frac{R_1 R_2 R_3}{R_2 R_3 + R_1 R_3 + R_1 R_2}$$

(the answer to Worked example 4.12) may be the simplest you can give. It can be very tempting to 'cancel' terms incorrectly in an attempt to get to the sort of simple fraction which is generally achievable when simplifying numerical fractions, but less likely to be achievable when dealing with symbols.

Question

Express $\dfrac{2c\sqrt{a}}{(a+2)} \times \dfrac{(b+2)}{2c\sqrt{b}}$ as a single fraction of the simplest possible form.

Answer

We can cancel the '2c's to give

$$\frac{\cancel{2c}\sqrt{a}}{(a+2)} \times \frac{(b+2)}{\cancel{2c}\sqrt{b}} = \frac{\sqrt{a}\,(b+2)}{(a+2)\sqrt{b}}$$
$$= \frac{\sqrt{a}\,(b+2)}{\sqrt{b}\,(a+2)}$$

It can be tempting to 'cancel' the square roots and the '+2's too, but this would be incorrect:

$$\frac{\sqrt{a}}{\sqrt{b}} \neq \frac{a}{b} \quad \text{and} \quad \frac{(b+2)}{(a+2)} \neq \frac{b}{a}$$

As discussed in Section 1.2.3, a fraction is unchanged by the multiplication or the division of both its numerator and denominator by the same amount. However, *all other operations will alter its value*.

So $\sqrt{\frac{a}{b}}\,\frac{(b+2)}{(a+2)}$ is as far as it is possible to simplify $\frac{2c\sqrt{a}}{(a+2)} \times \frac{(b+2)}{2c\sqrt{b}}$.

Note however that $\sqrt{\frac{a}{b}}$ is equivalent to $\frac{\sqrt{a}}{\sqrt{b}}$, so $\frac{\sqrt{a}}{\sqrt{b}}\,\frac{(b+2)}{(a+2)}$ can also be written as $\sqrt{\frac{a}{b}}\,\frac{(b+2)}{(a+2)}$.

Question 4.5

Simplify the following expressions, giving each answer as a single fraction.

(a) $\dfrac{\mu_0}{2\pi} \times \dfrac{i_1 i_2}{d}$ Answer

(b) $\dfrac{3a}{2b} \bigg/ 2$ Answer

(c) $\dfrac{2b}{c} + \dfrac{3c}{b}$ Answer

(d) $\dfrac{2ab}{c} \div \dfrac{2ac}{b}$ Answer

(e) $\dfrac{1}{f} - \dfrac{1}{f+1}$ Answer

(f) $\dfrac{2b^2}{(b+c)} \div \dfrac{2c^2}{(a+c)}$ Answer

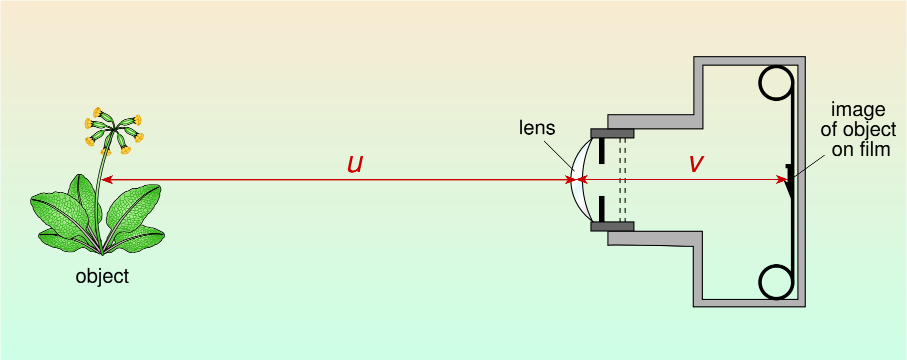


Figure 4.3: The object and image of a simple camera.

Question 4.6 Answer

The distance, u, of an object from a lens (such as the lens in the simple camera illustrated in Figure 4.3) is related to the distance, v, from the lens to the image of the object (on the film) and the lens's focal length, f, by the equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

Add the fractions $1/u$ and $1/v$ and hence rearrange the equation to give an expression for f.

4.2.2 Using brackets in algebra

You should be familiar by now with the notion that an operation applied to an expression in a bracket must be applied to *everything* within the bracket, so

$$(2b)^2 = 2^2b^2 = 4b^2$$
$$(a + 2b) - (a + b) = a + 2b - a - b = b$$
$$(a + 2b) - (a - b) = a + 2b - a - (-b) = a + 2b - a + b = 3b$$
$$2(a + 2b) = (2 \times a) + (2 \times 2b) = 2a + 4b$$

and

$$2a(a + 2b) = (2a \times a) + (2a \times 2b) = 2a^2 + 4ab$$

If we need to multiply two bracketed expressions, such as $(a+b)$ and $(c+d)$ together, we need to multiply *each* term in the first bracket by *each* term in the second bracket as indicated by the red lines shown below.

$$(a + b)(c + d)$$

Multiplying the terms in order gives

$$(a + b)(c + d) = \overset{1}{ac} + \overset{2}{ad} + \overset{3}{bc} + \overset{4}{bd}$$

Worked example 4.13

Rewrite the following expressions so that the brackets are removed:

(a) $(x-3)(x+5)$

(b) $(x+y)(x-y)$

(c) $(x+y)^2$

(d) $(x-y)^2$

Answer

(a) $$(x-3)(x+5) = x^2+5x-3x-15$$
$$= x^2+2x-15$$

(b) $$(x+y)(x-y) = x^2-xy+yx-y^2$$
$$= x^2-y^2 \quad \text{since } xy = yx, \text{ so } -xy+yx = 0$$

(c) $$(x+y)^2 = (x+y)(x+y)$$
$$= x^2+xy+yx+y^2$$
$$= x^2+2xy+y^2$$

(d) $$(x-y)^2 = (x-y)(x-y)$$
$$= x^2-xy-yx+y^2$$
$$= x^2-2xy+y^2$$

An examination of the answers to parts (b), (c) and (d) of Worked example 4.13 serves as a reminder of the fact that

$$(x+y)^2 \neq x^2 + y^2$$

$$(x-y)^2 \neq x^2 - y^2$$

In other words, remember to watch out for brackets!

Question 4.7

Rewrite the following expressions so that the brackets are removed:

(a) $\dfrac{1}{2}(v_x + u_x)\,t$ Answer

(b) $\dfrac{(a-b)-(a-c)}{2}$ Answer

(c) $(k-2)(k-3)$ Answer

(d) $(t-2)^2$ Answer

So far, this section has discussed removing brackets from expressions, but it can very often be useful to do the reverse.

The numbers 6 and 4 are described as factors of 24 and in general, when speaking mathematically, 'factors' are terms which when multiplied together give the original expression. Since, for example,

$$y\,(y+3) = y^2 + 3y$$

we can say that y and $(y + 3)$ are factors of $y^2 + 3y$

Similarly, since

$$\begin{aligned}(x+3)(x-1) &= x^2 - x + 3x - 3\\ &= x^2 + 2x - 3\end{aligned}$$

we can say that $(x + 3)$ and $(x - 1)$ are factors of $x^2 + 2x - 3$.

The verb 'to factorize' means to find the factors of an expression. If you are asked to factorize $y^2 + 3y$ then you should write

$$y^2 + 3y = y(y+3)$$

and if you are asked to factorize $x^2 + 2x - 3$ you should write

$$x^2 + 2x - 3 = (x+3)(x-1)$$

Note, from Worked example 4.13b, that the factors of $x^2 - y^2$ are $(x + y)$ and $(x - y)$, i.e.

$$x^2 - y^2 = (x + y)(x - y) \tag{4.3}$$

The difference of two squared numbers can always be written as the product of their sum and their difference.

Question 4.8

Factorize the following expressions:

(a) $y^2 - y$ Answer

(b) $x^2 - 25$ (*Hint*: you may find it helpful to compare this expression with Equation 4.3, remembering that $5^2 = 25$.) Answer

Factorizing can be useful when rearranging equations, as Worked example 4.14 illustrates.

Worked example 4.14

Rearrange $q = mc\,\Delta T + mL$ so that m is the subject.

Answer

Both terms on the right-hand side of this equation include m, so we can rewrite the equation as

$$q = m\,(c\,\Delta T + L)$$

This can be reversed to give

$$m\,(c\,\Delta T + L) = q$$

Now we divide both sides by $(c\,\Delta T + L)$ to give

$$m = \frac{q}{c\,\Delta T + L}$$

Question 4.9 Answer

Rearrange $E_{\mathrm{tot}} = \frac{1}{2}mv^2 + mg\,\Delta h$ to give an equation for m.

An ability to factorize expressions such as $y^2 + 3y$ and $x^2 + 2x - 3$ can also help us to find the solutions of equations such as $y^2 + 3y = 0$ and $x^2 + 2x - 3 = 0$. Equations of this form are known as 'quadratic equations'.

We know from above that

$$y^2 + 3y = y\,(y + 3) \tag{4.4}$$

So if $y^2 + 3y = 0$, it follows that $y\,(y + 3) = 0$ too. Multiplying by zero gives zero (as discussed in Section 1.1.4). So $y\,(y + 3) = 0$ implies that either $y = 0$ or $y + 3 = 0$.

$y + 3 = 0$ implies that $y = -3$, so the solutions of $y^2 + 3y = 0$ are $y = 0$ and $y = -3$.

We can check that $y = 0$ and $y = -3$ really are solutions of the equation $y^2 + 3y = 0$, by substituting each value for y into the left-hand side of the equation and verifying that it gives 0, as expected.

For $y = 0$, $y^2 + 3y = 0 + 0 = 0$, as expected.

For $y = -3$, $y^2 + 3y = (-3)^2 + \big(3 \times (-3)\big) = 9 + (-9) = 0$, as expected.

It is sensible to check your answers in this way:

> You should check your answers whenever possible.

Worked example 4.15

Use the fact that

$$x^2 + 2x - 3 = (x + 3)(x - 1) \tag{4.5}$$

to find the solutions of the equation $x^2 + 2x - 3 = 0$.

Answer

If $x^2 + 2x - 3 = 0$ then, from Equation 4.5, $(x + 3)(x - 1) = 0$

Thus $x + 3 = 0$ or $x - 1 = 0$, i.e. $x = -3$ or $x = 1$.

Checking for $x = -3$:

$$x^2 + 2x - 3 = (-3)^2 + 2(-3) - 3 = 9 - 6 - 3 = 0, \text{ as expected.}$$

Checking for $x = 1$:

$$x^2 + 2x - 3 = 1^2 + (2 \times 1) - 3 = 1 + 2 - 3 = 0, \text{ as expected.}$$

So the solutions of the equation $x^2 + 2x - 3 = 0$ are $x = -3$ and $x = 1$.

Using factorization to solve quadratic equations relies on us being able to spot the factors of an expression; this is quite easy for expressions like $y^2 + 3y$ (see Equation 4.4), but if we had not known or been told that $x^2 + 2x - 3 = (x + 3)(x - 1)$ (Equation 4.5), finding the factors of x^2+2x-3 would have been largely a matter of trial and error. An ability to find factors in this way can be developed with practice, but it remains somewhat tedious and this method for solving quadratic equations doesn't work at all unless the solutions are whole numbers or simple fractions. Fortunately help is at hand in the form of the 'quadratic equation formula', described in Box 4.4.

Box 4.4 The quadratic equation formula

Al-Khwarizmi and other early Arab mathematicians developed general methods for solving quadratic equations. A quadratic equation of the form

$$ax^2 + bx + c = 0$$

will have solutions given by the quadratic equation formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 > 4ac$ (i.e. b^2 is greater than $4ac$) then $b^2 - 4ac$ will be positive, and the formula will lead to two distinct solutions.

If $b^2 = 4ac$ then $b^2 - 4ac = 0$, so the two solutions will be identical ($x = -b/(2a)$).

If $b^2 < 4ac$ (i.e. b^2 is less than $4ac$) then $b^2 - 4ac$ will be negative. This means that the solutions will include the square root of a negative number. and hence will involve ‘imaginary numbers’. Such numbers were mentioned in Chapter 1, but will not be considered further in *Maths for Science*.

Worked example 4.16 demonstrates the use of the quadratic equation formula in solving the equation that was solved by factorization in Worked example 4.15.

Worked example 4.16

Use the quadratic equation formula to find the solutions of the equation $x^2 + 2x - 3 = 0$.

Answer

Comparison of

$$x^2 + 2x - 3 = 0$$

and

$$ax^2 + bx + c = 0$$

shows that $a = 1$, $b = 2$ and $c = -3$ on this occasion, so the solutions are

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-2 \pm \sqrt{2^2 - \big(4 \times 1 \times (-3)\big)}}{2 \times 1} \\
&= \frac{-2 \pm \sqrt{4 - (-12)}}{2} \\
&= \frac{-2 \pm \sqrt{16}}{2} \\
&= \frac{-2 \pm 4}{2}
\end{aligned}$$

So $x = \dfrac{-2 + 4}{2} = 1$

or $x = \dfrac{-2 - 4}{2} = -3$

The solutions can be checked in exactly the same way as in Worked example 4.15.

Once again, we have found that the solutions of the equation $x^2 + 2x - 3 = 0$ are $x = -3$ and $x = 1$.

Question 4.10

(a) Use your answer to Question 4.7 (c) to solve $k^2 - 5k + 6 = 0$ by factorization. Answer

(b) Use your answer to Question 4.7 (d) to solve $t^2 - 4t + 4 = 0$ by factorization. Answer

(c) Use the quadratic equation formula to check your answer to part (a). Answer

(d) Use the quadratic equation formula to check your answer to part (b). Answer

4.3 Combining equations

Consider the equation $E = hf$. This equation, first proposed by Einstein, links the energy, E, of light to its frequency, f (h is a constant known as Planck's constant). Suppose that you know h and are trying to find E, but that you don't know f. Instead you know the values of c (speed of light) and λ (wavelength) in a second equation, $c = f\lambda$. It would be possible to calculate a value for f from the second equation and then substitute this value in the first equation so as to find E. However, this approach runs the risk of numerical slips and rounding errors. It is more useful to do the substitution *algebraically*, in the way shown in the following example.

Worked example 4.17

Combine the following two equations to find an equation for E not involving f:

$$E = hf \tag{4.6}$$

$$c = f\lambda \tag{4.7}$$

Answer

Rearranging Equation 4.7 gives

$$f = \frac{c}{\lambda}$$

Substituting this expression for f into Equation 4.6 gives

$$E = h \times \frac{c}{\lambda} = \frac{hc}{\lambda}$$

This mathematical technique, sometimes referred to as elimination (because a variable, f on this occasion, is being eliminated), can be used in many situations, as illustrated in the worked examples throughout this section.

Worked example 4.18

Combine $F_\mathrm{g} = G\dfrac{Mm}{r^2}$ and $F_\mathrm{g} = mg$ to give an equation for r not involving F_g.

Answer

Since both equations are already given with F_g (the variable we are trying to eliminate) as the subject, we can simply set the two equations for F_g equal to each other:

$$mg = G\frac{Mm}{r^2}$$

We now need to rearrange to give an equation for r. First note that there is an m on both sides of the equation, so we can divide both sides of the equation by m to give

$$g = G\frac{M}{r^2}$$

Multiplying both sides by r^2 gives

$$gr^2 = GM$$

Dividing both sides by g gives

$$r^2 = \frac{GM}{g}$$

Taking the square root of both sides gives

$$r = \pm\sqrt{\frac{GM}{g}}$$

Question 4.11

(a) Combine $E_{\mathrm{k}} = \frac{1}{2}mv^2$ and $p = mv$ to give an equation for E_{k} not involving v. Answer

(b) Combine $E = \frac{1}{2}mv^2$ and $E = mg\,\Delta h$ to give an equation for v not involving E. Answer

(c) Combine $E_{\mathrm{k}} = hf - \phi$ and $c = f\lambda$ to give an equation for ϕ not involving f. Answer

Two (or more) different equations containing the same two (or more) unknown quantities are called 'simultaneous equations' if the equations must be satisfied (hold true) simultaneously. It is usually possible to solve two simultaneous equations by using one equation to eliminate one of the unknown quantities from the second equation, in an extension of the method discussed above. This is illustrated in Worked example 4.19.

Worked example 4.19

Find values of x and y which satisfy both of the equations given below:

$$x + y = 7 \tag{4.8}$$

$$2x - y = 2 \tag{4.9}$$

Answer

If we rewrite Equation 4.8 to give an equation for y in terms of x, then we can insert this result into Equation 4.9 to give an equation for x alone.

Subtracting x from both sides of Equation 4.8 gives

$$y = 7 - x \tag{4.10}$$

Substituting for y in Equation 4.9 then gives

$$\begin{aligned} & 2x - (7 - x) = 2 \\ \text{i.e.}\ & 2x - 7 + x = 2 \\ \text{or}\ \ & 3x - 7 = 2 \end{aligned}$$

Adding 7 to both sides gives

$$3x = 9, \text{ i.e. } x = 3$$

Substitution of $x = 3$ into Equation 4.10 shows that

$$y = 7 - x = 7 - 3 = 4$$

So the solution (i.e. the values for x and y for which both of the equations hold true) is $x = 3$ and $y = 4$. We can check this by substituting the values for x and y into the left-hand side of Equations 4.8 and 4.9.

Equation 4.8 gives $x + y = 3 + 4 = 7$, as expected.
Equation 4.9 gives $2x - y = (2 \times 3) - 4 = 6 - 4 = 2$, as expected.

We could have arrived at the same result by using Equations 4.8 and 4.9 in a different order, but there is only one correct answer.

Worked example 4.19 shows that in order to find two unknown quantities, two different equations relating them are required. This is always true and by extension:

When you combine equations so as to find unknown quantities, it is always necessary to have at least as many equations as there are unknown quantities.

Worked example 4.20 shows how four equations can be combined together in a case where there are four unknown quantities (we are trying to find the total surface area, S, but the mass, m, and volume, V, of a single particle and the number of particles, n, are unknown too and so must be eliminated). This worked example concerns the use of metal particles as catalysts in the chemical industry (see Box 4.5).

Box 4.5 Chemical catalysts

A catalyst is a substance which makes a chemical reaction proceed more rapidly. The catalyst itself does not undergo permanent chemical change, and it can be recovered when the chemical reaction is completed. Metal particles can be used as catalysts. A large number of small particles will have a greater surface area than a small number of larger particles, and the total surface area, S, of the particles is of critical importance to the speed of the reaction. In a typical industrial chemical reactor, S can be approximately 5000 km^2; roughly a third the area of Yorkshire!

Worked example 4.20

The total surface area, S, of n metal particles of average radius r is given by the equation

$$S = 4\pi nr^2 \tag{4.11}$$

The number of particles n is linked to the mass of one particle, m and the total mass of metal, M by the equation

$$n = \frac{M}{m} \tag{4.12}$$

The mass m of one particle is linked to its volume V and the density of the metal ρ by the equation

$$\rho = \frac{m}{V} \tag{4.13}$$

The volume V of a particle is given by

$$V = \frac{4}{3}\pi r^3 \tag{4.14}$$

where r is the radius.

Find an equation for S in terms of M, ρ and r only.

Answer

Reversing Equation 4.13 gives

$$\frac{m}{V} = \rho$$

Multiplying both sides by V gives

$$m = V\rho$$

Substituting for V from Equation 4.14 gives

$$m = \frac{4}{3}\pi r^3\rho$$

Substituting this expression for m into Equation 4.12 gives

$$\begin{aligned} n &= \frac{M}{m} \\ &= \frac{M}{\frac{4}{3}\pi r^3\rho} \\ &= \frac{3M}{4\pi r^3\rho} \end{aligned}$$

Substituting this expression for n into Equation 4.11 gives

$$\begin{aligned} S &= 4\pi n r^2 \\ &= \cancel{4\pi} \times \frac{3M}{\cancel{4\pi} r^{\cancel{3}}\rho} \times \cancel{r^2} \\ &= \frac{3M}{r\rho} \end{aligned}$$

4.4 Putting algebra to work

So far, Chapter 4 has been concerned almost exclusively with symbols. Equations have been given to you and you have been told to manipulate them in a particular way. In the real scientific world, you are likely to need to:

1 Choose the correct equation(s) to use or derive equation(s) for yourself.

2 Combine, rearrange and simplify the equation(s) using the skills introduced in the earlier sections of this chapter.

3 Substitute numerical values, taking care over things like significant figures, scientific notation and units, as you did in Chapter 3.

4 Check that the answer is reasonable.

The final section of this chapter considers these points, combining skills from Chapters 3 and 4, but it starts with a more light-hearted look at the uses of algebra.

4.4.1 Algebra is fun!

Try this:

- Think of a number.
- Double it.
- Add four.
- Halve your answer.
- Subtract 1.

If you have arrived at an answer of 4, I can tell you that the number you first thought of was 3; if your answer is 6, the number you first thought of was 5, if your answer is 11, the number you first thought of was 10, and so on. Magic? No, a demonstration of the power of algebra! We could perform exactly the same operations for *any* number; let's represent the number by the symbol N. Then we have

• Think of a number.	N
• Double it.	$2N$
• Add four.	$2N + 4$
• Halve your answer.	$\frac{1}{2}(2N + 4) = N + 2$
• Subtract 1.	$(N + 2) - 1 = N + 1$

So the final answer will always be one more than the number you first thought of.

Here's another one for you to try:

- Think of a number.
- Add 5.
- Double the result.
- Subtract 2.
- Divide by 2.
- Take away the number you first thought of.

Whatever number you first thought of, the answer will always be four.

Question 4.12 Answer

Use a symbol of your choice to represent the number in the 'think of a number' example immediately before this question and thus show that the answer will be four, whatever number you choose at the beginning.

You may wonder why a course entitled *Maths for Science* has suddenly started discussing number tricks. There is a serious point to this, namely to illustrate how you can get from an initial problem to a solution by using algebra. Worked example 4.21 illustrates another use of algebra.

Worked example 4.21

Chris and Jo share a birthday (but are different ages). On their birthday this year Chris will be five times older than Jo. Their combined age on their birthday last year was 58. How old was Chris when Jo was born?

Answer

Let C represent Chris's age in years on her birthday this year and J represent Jo's age in years on her birthday this year.

Since Chris will be five times older than Jo we can say

$$C = 5J \tag{4.15}$$

Last year Chris's age was $(C - 1)$ and Jo's age was $(J - 1)$, so we can say

$$\begin{aligned} & (C - 1) + (J - 1) = 58 \\ \text{i.e. } & C + J - 2 = 58 \\ & C + J = 60 \end{aligned} \tag{4.16}$$

Substituting for C from Equation 4.15 in Equation 4.16 gives

$$\begin{aligned} & 5J + J = 60 \\ \text{i.e. } & 6J = 60 \\ & J = 10 \end{aligned}$$

Thus, from Equation 4.15, $C = 5 \times 10 = 50$.

Thus Chris will be 50 this year and Jo will be 10. But this wasn't the question that was asked! When Jo was born, Chris was 50 – 10, i.e. 40 years old.

You may remember questions like Worked example 4.21 from your school days. Problems like this can seem intimidating, but they are relatively easy to solve once you have found the equations that describe the problem. Many people struggle with this first step — they can't find the equations to use. Look at Worked example 4.21 carefully; all that has been done in order to derive Equation 4.15 and Equation 4.16 has been to study carefully the information given in the question, and to write it down in terms of symbols. So 'On their birthday this year Chris will be five times older than Jo' has become $C = 5J$. In solving problems, it is almost always helpful to start by writing down what you already know. Drawing a diagram to illustrate the situation can help too; you may find this helpful in Question 4.13.

Question 4.13 Answer

Tracey is 15 cm taller than Helen, and when Helen stands on Tracey's shoulder she can just see over a fence 3 m tall. Assume that it is 25 cm from Tracey's shoulder to the top of her head and 10 cm from Helen's eyes to the top of her head. How tall is Helen?

4.4.2 Using algebra to solve scientific problems

In much the same way as people struggle when trying to derive equations for use in problems like Worked example 4.21, they often have difficulty deciding which formulae to use from those given in a book or on a formula sheet. Again, it can be helpful to draw a diagram and it is *always* helpful to start by writing down what you know and what you're trying to find. This often helps you to decide how to proceed.

Worked example 4.22 discusses the choice of appropriate formulae for use in answering a particular question. It also works through the other steps you are likely to follow when using algebra to solve scientific problems.

Worked example 4.22

A silver sphere (density $10.49\ \mathrm{g\,cm^{-3}}$) has a radius of 2.5 mm. What is its mass? Use formulae given in Box 3.4.

Which equations shall we use?

We know density (ρ) and radius (r) and are trying to find mass (m), so we need an equation to link these three variables. Equation 3.9, $\rho = \dfrac{m}{V}$, links density and mass, but it also includes *volume* which isn't either given or required by the question. Fortunately help is at hand in the form of Equation 3.5, $V = \frac{4}{3}\pi r^3$ which gives the volume V of a sphere of radius r. We should be able to substitute for V from Equation 3.5 into Equation 3.9. This will give an equation involving only ρ, r and m, as required, and we can then rearrange it to make m the subject.

Combining and rearranging equations

Substituting for V from Equation 3.5 into Equation 3.9 gives

$$\rho = \frac{m}{\frac{4}{3}\pi r^3}$$

Multiplying top and botom of the fraction by 3 gives

$$\rho = \frac{3m}{4\pi r^3}$$

Reversing this so that m is on the left-hand side gives

$$\frac{3m}{4\pi r^3} = \rho$$

Multiplying both sides by $4\pi r^3$ gives

$$3m = 4\pi r^3 \rho$$

Dividing both sides by 3 gives

$$m = \frac{4}{3}\pi r^3 \rho$$

Substituting numerical values

Note that we have used symbols for as long as possible in this question, so as to avoid numerical slips and rounding errors. However, we are now almost ready to substitute the values given for r and ρ. First we need to convert the values given into consistent (preferably SI) units:

$$r = 2.5\text{ mm} = 2.5 \times 10^{-3}\text{ m}$$

$\rho = 10.49\text{ g cm}^{-3} = 10.49 \times 10^{3}\text{ kg kg}^{-3}$ $(1.049 \times 10^{4}\text{ kg kg}^{-3}$ in scientific notation), converting from g cm^{-3} to kg m^{-3} in the way described in Section 3.4.4. Then

$$\begin{aligned} m &= \frac{4}{3}\pi r^3 \rho \\ &= \frac{4}{3}\pi \left(2.5 \times 10^{-3}\text{ m}\right)^3 \times 1.049 \times 10^{4}\text{ kg kg}^{-3} \\ &= 6.9 \times 10^{-4}\ \cancel{\text{m}^3}\text{ kg}\ \cancel{\text{m}^{-3}} \\ &= 6.9 \times 10^{-4}\text{ kg} \end{aligned}$$

Is the answer reasonable?

It is always worth spending a few minutes checking whether the answer you have arrived at is reasonable. There are three simple ways of doing this (it is not normally necessary to use all three methods to check one answer):

1 We can check the units of the answer. We have given units next to all the numerical values in the calculation, and the units on the right-hand side of the equation have worked out to be kilograms, as we would expect for mass.

If we had made a mistake in transposing the formula for mass, and had written it as $m = \frac{4}{3}\pi r^2 \rho$ by mistake, then the units on the right-hand side would have been $\mathrm{m}^2 \times \mathrm{kg\,m}^{-3} = \mathrm{kgm}^{-1}$. These are not units expected for mass by itself, so we would have been alerted to the fact that something was wrong.

Checking units in this way provides a good way of checking that you have written down or derived an equation correctly; the units on the left-hand side of an equation should always be equal to the units on the right-hand side. You can use this method for checking an equation even if you are not substituting numerical values into it.

2 We can estimate the value (in the way described in Section 3.3), and compare it with the answer found on a calculator. In this case

$$
\begin{aligned}
m &\approx \frac{4}{3} \times 3\left(3 \times 10^{-3}\ \mathrm{m}\right)^3 \times 1 \times 10^4\ \mathrm{kg\,m}^{-3} \\
&\approx \frac{4}{\cancel{3}} \times \cancel{3} \times 3^3 \times 10^{-9}\ \cancel{\mathrm{m}^3} \times 1 \times 10^4\ \mathrm{kg}\,\cancel{\mathrm{m}^{-3}} \\
&\approx 4 \times 27 \times 10^{-9+4}\ \mathrm{kg} \\
&\approx 100 \times 10^{-5}\ \mathrm{kg} \\
&\approx 1 \times 10^{-3}\ \mathrm{kg}
\end{aligned}
$$

This is the same order of magnitude as the calculated value, so the calculated value seems reasonable.

3 We can look at the answer and see if it is what common sense might lead us to expect. Obviously this method only works when you are doing a calculation concerning physical objects with which you are familiar, but it gives a sensible check for worked examples like the one we are considering. It seems reasonable that a silver sphere with a diameter of 0.5 cm might have a mass of something less than a gram. If you'd arrived at an answer of 1.1×10^2 kg (by forgetting to cube the value given for r) you might have thought that this mass (equivalent to more than 100 bags of sugar!) was rather large for such a small sphere.

Note that checking doesn't usually tell you that your answer is absolutely correct — none of the methods described above would have spotted small arithmetic slips — but it does frequently alert you if the answer is wrong.

Tips for using algebra to solve scientific problems

1 Start by writing down what you know and what you're trying to find, and use this information to find appropriate equations to use.

2 Combine, rearrange and simplify the equations, using symbols for as long as possible so as to avoid numerical slips and rounding errors.

3 When you substitute numerical values, take care with units, scientific notation and significant figures.

4 Check that your final answer is reasonable, by asking yourself the following questions:

  (a) Are the units what you would expect?

  (b) Is the answer similar to the one you have obtained by estimating?

  (c) Is the answer about what you would expect from common sense?

Worked example 4.23 shows the use of these tips in solving a different problem, concerning the conservation of energy. This worked example uses formulae introduced in Box 4.6; you may also find these formulae useful when answering Question 4.14.

Box 4.6 The conservation of energy

Energy can never be destroyed, but it is frequently converted from one form to another. As a child climbs the steps of a slide, he or she gains in gravitational potential energy; as he or she slides down the slide this energy is converted into kinetic (movement) energy. As a kettle boils, the electrical energy increases the energy of the water molecules and so raises the temperature of the water. In both cases some energy is 'lost' to other forms (such as heat to the surroundings and sound) but very often you can assume that all of the energy initially in one form is converted to just one other form, and so equate formulae (such as those given below) for different forms of energy. All forms of energy should be quoted using the SI unit of energy which is the joule (J), where $1\ \mathrm{J} = 1\ \mathrm{kg\,m^2\,s^{-2}}$.

The kinetic energy (energy of motion), E_k, of an object with a mass m moving at speed v is given by

$$E_\mathrm{k} = \tfrac{1}{2}mv^2 \tag{4.17}$$

The gravitational potential energy, E_g, of an object of mass m at a height Δh above a reference level is given by

$$E_\mathrm{g} = mg\,\Delta h \tag{4.18}$$

where g is the acceleration due to gravity.

The energy, q, needed to raise the temperature of a mass m of a substance of specific heat capacity c by a temperature ΔT is given by

$$q = mc\,\Delta T \tag{4.19}$$

Worked example 4.23

A lump of putty is dropped from a height of 4.8 m. The putty's gravitational potential energy is all converted into kinetic energy as it falls. If, on impact, all of this kinetic energy is used to raise the temperature of the putty, by how much does the temperature of the putty rise? Assume that the specific heat capacity of the putty is $5.0 \times 10^2\ \mathrm{J\,kg^{-1}\,K^{-1}}$ and that the acceleration due to gravity is $9.81\ \mathrm{m\,s^{-2}}$.

Which equations shall we use?

It is tempting to involve Equation 4.17, as the question talks about the putty's kinetic energy, but closer inspection of the question reveals that we can assume that all the gravitational potential energy becomes kinetic energy as the putty falls, and that all the kinetic energy is transferred to heat energy in the putty on impact. So we can say that all the gravitational potential energy is transferred to heat energy; we simply need to set Equations 4.18 and 4.19 equal to each other. We have not been told the mass of the putty, but since the term m appears in both Equation 4.18 and Equation 4.19 we will be able to cancel this term, which will leave us with an equation linking g, Δh, c and ΔT. We know g, Δh and c and are trying to find ΔT.

Combining and rearranging equations

Since we can assume that all the gravitational potential energy, E_g, is transferred to heat energy, q, we can set Equation 4.18 and Equation 4.19 equal to each other.

$$mc\,\Delta T = mg\,\Delta h$$

There is an m on both sides, so we can divide by m to give

$$c\,\Delta T = g\,\Delta h$$

Dividing both sides by c gives

$$\Delta T = \frac{g\,\Delta h}{c}$$

Substituting numerical values

$g = 9.81\ \mathrm{m\,s^{-2}}$
$h = 4.8\ \mathrm{m}$
$c = 5.0 \times 10^2\ \mathrm{J\,kg^{-1}\,K^{-1}}$

so

$$\begin{aligned}\Delta T &= \frac{g\,\Delta h}{c}\\ &= \frac{9.81\ \mathrm{m\,s^{-2}} \times 4.8\ \mathrm{m}}{5.0\times10^{2}\ \mathrm{J\,kg^{-1}\,K^{-1}}}\\ &= \frac{9.81\times4.8\ \cancel{\mathrm{m}}\,\cancel{\mathrm{s^{-2}}}\times\cancel{\mathrm{m}}}{5.0\times10^{2}\ \cancel{\mathrm{kg}}\,\cancel{\mathrm{m^{2}}}\,\cancel{\mathrm{s^{-2}}}\,\cancel{\mathrm{kg^{-1}}}\,\mathrm{K^{-1}}}\\ &= 0.094\ \mathrm{K}\ \text{to two significant figures.}\end{aligned}$$

Is the answer reasonable?

In a real question you probably wouldn't use all the checks described in the blue-toned box after Worked example 4.22, but the answer seems about the size you might expect (you wouldn't expect a big temperature rise) and the units have worked out to be kelvin, as expected for a change in temperature.

Alternatively we can estimate the answer to be

$$\Delta T \approx \frac{10\ \mathrm{m\,s^{-2}}\times5\ \mathrm{m}}{5\times10^{2}\ \mathrm{J\,kg^{-1}\,K^{-1}}} \approx 10^{-1}\ \mathrm{K}$$

This is the same order of magnitude as the calculated value, so the calculated value seems reasonable.

Question 4.14 Answer

A child climbs to the top of a 1.8 m slide and then slides to the ground. Assuming that all of her gravitational potential energy is converted into kinetic energy, find her speed as she reaches the ground. Take $g = 9.81\ \mathrm{m\,s^{-2}}$ and use appropriate formulae from Box 4.6.

In Worked example 4.24, the final worked example in Chapter 4, we return to a discussion of seismic waves travelling through the Earth's crust (as introduced in Box 3.1). In this example there are three unknown quantities (the distance, d, from the earthquake, the time, t_P, taken for P waves to reach the seismometer and the time, t_S, taken for S waves to reach the seismometer) so we need to combine three equations to find any of the unknown quantities. You will not be expected to combine more than two equations together in any questions associated with this course, but Worked example 4.24 has been included because it summarizes much of what has been discussed in Chapter 4, and also because it illustrates the usefulness of algebra in science.

Box 4.7 Locating an earthquake

Figure 4.4 shows a seismogram recorded at the British Geological Survey in Edinburgh on 12 September 1988. It is possible to see the points at which P waves and S waves first reached the seismometer. We can assume that these seismic waves originated in an earthquake somewhere. But where was the earthquake and when did it occur? (although the recording was made at 2.23 p.m., it does not tell us the time at which the earthquake occurred, since the waves will have taken some time to reach the seismometer from the point of origin or focus of the earthquake).

Figure 4.4 shows that the P waves reached the seismometer 20 seconds before the S waves.

We assume that the P waves travelled with an average speed, $v_p = 5.6\ \mathrm{km\,s^{-1}}$ and that the S waves travelled with an average speed $v_s = 3.4\ \mathrm{km\,s^{-1}}$ (these values are typical for the rocks of the Earth's crust, through which the waves will have been travelling).

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

$$\text{so} \qquad v_p = \frac{d}{t_p} \tag{4.20}$$

$$\text{and} \qquad v_s = \frac{d}{t_s} \tag{4.21}$$

where d is the distance from the earthquake, t_p is the time taken for P waves to travel to the seismometer and t_s is the time taken for S waves to travel to the seismometer.

Worked example 4.24

Use the information given in Box 4.7 to find the distance from Edinburgh to the focus of the earthquake recorded on the seismogram shown in Figure 4.4.

Which equations shall we use?

We know that $v_\text{p} = \dfrac{d}{t_\text{p}}$ (Equation 4.20) and $v_\text{s} = \dfrac{d}{t_\text{s}}$ (Equation 4.21), where $v_\text{p} = 5.6\ \text{km s}^{-1}$ and $v_\text{s} = 3.4\ \text{km s}^{-1}$, but d, t_p and t_s are all unknown, so we need another equation.

Although we don't know the travel time of the two types of wave, we know that the difference in the arrival time of the two waves is 20 seconds, so we can write

$$t = t_\text{s} - t_\text{p} \tag{4.22}$$

where $t = 20$ s.

Equations 4.20, 4.21 and 4.22 give us three equations containing the three unknowns d, t_p and t_s and we need to combine and rearrange them to give an expression for d.

Combining and rearranging equations

Multiplying both sides of Equation 4.20 by t_p gives

$$t_\text{p} v_\text{p} = d$$

Dividing both sides by v_p gives

$$t_\mathrm{p} = \frac{d}{v_\mathrm{p}}$$

Similarly, from Equation 4.21,

$$t_\mathrm{s} = \frac{d}{v_\mathrm{s}}$$

Substituting for t_s and t_p in Equation 4.22 gives

$$\begin{aligned} t &= t_\mathrm{s} - t_\mathrm{p} \\ &= \frac{d}{v_\mathrm{s}} - \frac{d}{v_\mathrm{p}} \\ &= d\left(\frac{1}{v_\mathrm{s}} - \frac{1}{v_\mathrm{p}}\right) \end{aligned}$$

Combining the fractions by making $v_\mathrm{s}v_\mathrm{p}$ a common denominator (Section 4.2.1) gives

$$t = d\,\frac{(v_\mathrm{p} - v_\mathrm{s})}{v_\mathrm{s}v_\mathrm{p}}$$

Reversing the equation so that d is on the left-hand side gives

$$d\,\frac{(v_\mathrm{p} - v_\mathrm{s})}{v_\mathrm{s}v_\mathrm{p}} = t$$

Multiplying both sides by $v_s v_p$ gives

$$d\,(v_p - v_s) = t\,v_s v_p$$

Dividing both sides by $(v_p - v_s)$ gives

$$d = \frac{t\,v_s v_p}{v_p - v_s}$$

Substituting numerical values

Substituting $t = 20$ s, $v_p = 5.6\ \mathrm{km\,s^{-1}}$ and $v_s = 3.4\ \mathrm{km\,s^{-1}}$ gives

$$\begin{aligned} d &= \frac{20\ \mathrm{s} \times 3.4\ \mathrm{km\,s^{-1}} \times 5.6\ \mathrm{km\,s^{-1}}}{(5.6\ \mathrm{km\,s^{-1}} - 3.4\ \mathrm{km\,s^{-1}})} \\ &= \frac{20\ \mathrm{s} \times 3.4\ \mathrm{km\,s^{-1}} \times 5.6\ \mathrm{km\,s^{-1}}}{2.2\ \mathrm{km\,s^{-1}}} \\ &= 1.7 \times 10^2\ \mathrm{km}\ \text{to two significant figures} \end{aligned}$$

The units work out to be kilometres since $\dfrac{\cancel{\mathrm{s}} \times \mathrm{km}\,\cancel{\mathrm{s}^{-1}} \times \cancel{\mathrm{km}}\,\cancel{\mathrm{s}^{-1}}}{\cancel{\mathrm{km}}\,\cancel{\mathrm{s}^{-1}}} = \mathrm{km}$

Is the answer reasonable?

The units have worked out to be kilometres as expected for a distance. If we had converted the speeds to values in $\mathrm{ms^{-1}}$, we would have obtained a value for d in metres ($d = 1.7 \times 10^5$ m).

In this case it is easy to check that the answer is reasonable; many members of the public reported a small earthquake on that day in Ambleside in Cumbria. Ambleside is 170.5 km from Edinburgh!

In general, to use this method to uniquely identify the location of an earthquake you need to repeat the exercise using data received at other seismometers elsewhere on the Earth's surface.

4.5 Learning outcomes for Chapter 4

After completing your work on this chapter you should be able to:

4.1 demonstrate understanding of the terms emboldened in the text;

4.2 rearrange an algebraic equation to make a different variable the subject;

4.3 simplify an algebraic expression;

4.4 add, subtract, multiply and divide algebraic fractions;

4.5 re-write an algebraic expression so that the brackets are removed;

4.6 factorize a simple algebraic expression;

4.7 eliminate one or more variables so as to combine equations together;

4.8 check the answer to a problem by checking units, estimating an answer, or comparing the answer with what would be expected from common sense.

5 Using Graphs

The well-known saying that a picture is worth a thousand words reflects the fact that human beings can derive a lot of information from pictorial representations of a situation. When scientists want to condense data into a visual form that conveys information at a glance, they most often turn to a graphical representation. Graphs are essential tools for scientific work: they can illustrate clearly the nature of the relationship between different quantities, they make it easy to see variations and trends and sometimes they can be used to derive other interesting quantities or even equations.

This chapter is mainly about the use and interpretation of graphs, rather than techniques for plotting them (which are more the province of courses in practical science). However, an understanding of the kind of information that can be derived from different types of graph will be of considerable help when you do come to plot your own data in the future.

5.1 Graphical representations

Although all graphs share certain characteristics, there are nevertheless a number of different ways in which data may be presented graphically. Let us start by considering some specific examples and the features they illustrate.

5.1.1 Bar charts and histograms

'Bar charts' are commonly used to summarize data that require immediate comparison between various discrete categories. Examples of discrete categories are human eye colour, blood group, countries and planets. The categories are listed along a reference line, usually a horizontal one (the so-called horizontal axis). The number or percentage of things or events falling into each category is represented by a bar; the scale for these bars, most commonly expressed either as a number or a percentage, is given on a second reference line, at right angles to the first. If the categories are listed along the horizontal axis, the bars will therefore be scaled along the vertical axis. Figure 5.1 in Box 5.1 is an example of how ecological data might be presented in the form of a bar chart.

Box 5.1 Insects and trees

Figure 5.1 shows the number of species of herbivorous insects associated with eight different types of native and introduced tree. Tree species that have been present in the country for a long time and are widely distributed often support the largest variety of insects.

Question

Roughly how many species of insect are associated with hawthorn?

Answer

About 220.

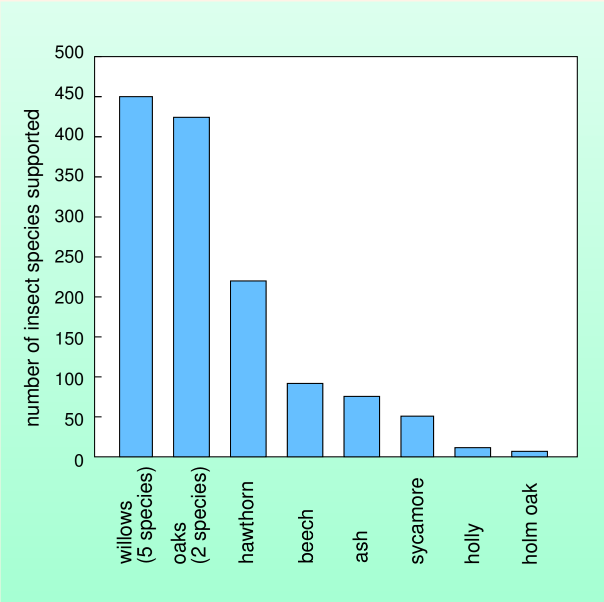


Figure 5.1: Bar chart showing the number of herbivorous insect species supported by some native and introduced tree species in the UK.

The willows and oaks, which are among the commonest tree species in the UK, can support over 400 insect species. Sycamore, which is just as widely distributed but came to this country more recently, supports only around 50 species, and the evergreen holm oak, which was introduced a mere 400 years ago, supports fewer than 10 insect species. However, one should not generalize too far from these examples. There are other native trees, such as holly and yew, which also support very few insect species, many of which are specialist feeders not found on other trees.

A histogram is similar to a bar chart in that numbers or percentages are again commonly plotted vertically, but on a histogram the horizontal axis is used to represent a continuously variable quantity such as height or mass. The purpose of a histogram is to show how the data are distributed into groups across a continuous range. Figure 5.2 shows a histogram which presents the results of measurements taken of the height of 100 irises. In principle, a plant selected at random could be of any height. Of those measured, a few specimens are particularly tall and a few are particularly short, but the majority are of intermediate height. This is typical of the natural variation in populations, and Chapters 8 and 9 deal with the statistical techniques that are required to analyse such variations. Comparing Figure 5.2 with Figure 5.1, you will notice that on the bar chart the bars do not touch (because they refer to different categories), whereas on the histogram the columns do touch, because all possible heights are represented within the groups marked on the horizontal axis. In Figure 5.2 the groups are of equal intervals, and this is common practice (though there are also ways of constructing histograms using unequal intervals). Note that the whole range of possible heights is covered, whether or not any of the measured plants actually fell into a particular group.

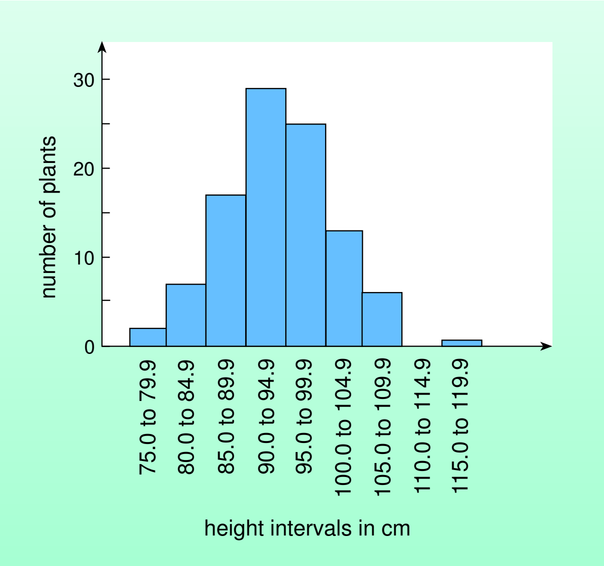


Figure 5.2: Histogram representing the heights of 100 of the same variety of iris. The horizontal axis is divided into intervals to represent different height groups.

5.1.2 Graphs

On a histogram, the horizontal axis is divided into intervals. On a graph, in contrast, the horizontal axis is scaled to represent a continuum. In Figure 5.3, for example, time is plotted along the horizontal axis, with the years being evenly spaced. This graph clearly shows the large variation in caterpillar numbers that can occur from year to year, though no overall trend can be discerned. It is not necessary to join the data points on a graph of this type; if this is done, as here, the lines have no significance beyond simply emphasizing the downturn or upturn in the numbers between one year and the next.

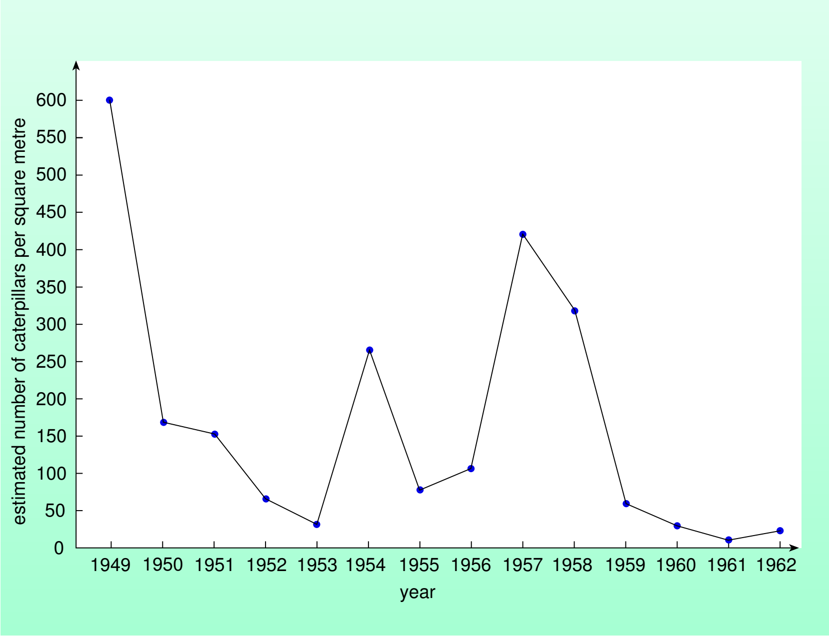
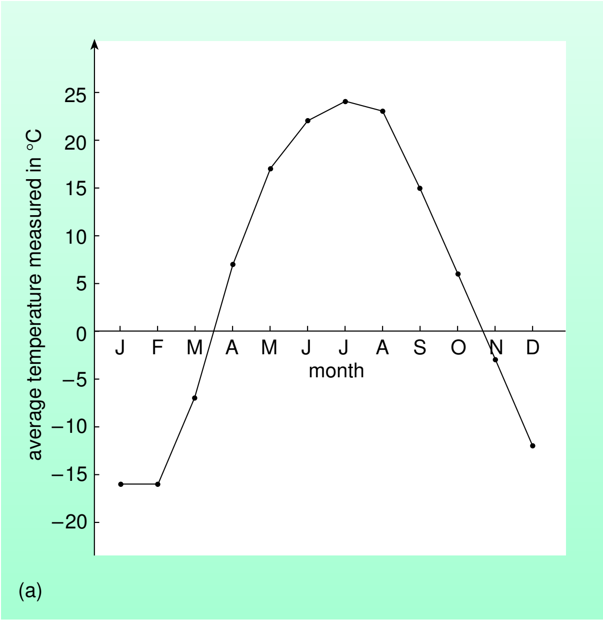


Figure 5.3: Annual fluctuations in the population of Winter Moth (*Operophtera brumata*) caterpillars feeding on oak trees in Wytham Wood near Oxford.



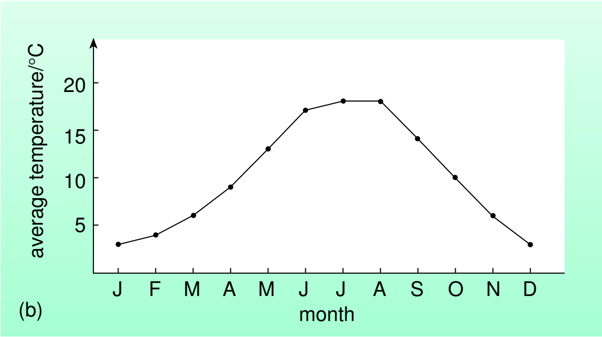


Figure 5.4: Average monthly temperatures for (a) Irgiz, Kazakhstan, and (b) Paris.

Figure 5.4a illustrates that negative, as well as positive numbers can be plotted on a graph; in this case the vertical axis covers temperatures from −20 °C to +25 °C. The data points have been joined, but the lines are only indicators of rises or falls in average temperature between one month and the next; they could not be used to predict the temperature on any particular day.

The vertical axis of Figure 5.4a is labelled to show that the quantity plotted is the average temperature measured in °C. Whatever the variable we want to display graphically, we always have to take account of its units in such a way as to plot a *pure number* (i.e. a number without units) on the graph. The labelling on the vertical axis of Figure 5.4a could have been written more succinctly as 'average temperature/°C', and this form of labelling has been used on the vertical axis in Figure 5.4b. The temperature values are divided by their unit (°C), to give pure numbers that can be plotted on the graph:

e.g. $\dfrac{2.3\ \cancel{°C}}{\cancel{°C}} = 2.3$

It is conventional always to use 'quantity divided by its units' (usually in the form 'quantity/units') in labelling the axes of graphs.

Box 5.2 Atmospheric pollution

In industrialized countries, air pollution was historically associated mainly with emissions of smoke and sulphur dioxide arising from the combustion of fossil fuels (chiefly coal) for domestic heating and industrial purposes. The resulting 'smogs' that occurred in Northern European cities for several centuries were the result of this kind of pollution. The problem became particularly acute in London in the 1950s, leading to the UK Clean Air Act of 1956. Subsequent European directives have further reduced emission limits and national emissions of sulphur dioxide have fallen dramatically — by about 80% since 1962. Figure

5.5 shows air quality data recorded in the Tameside district of Greater Manchester between 1963 and 2000.

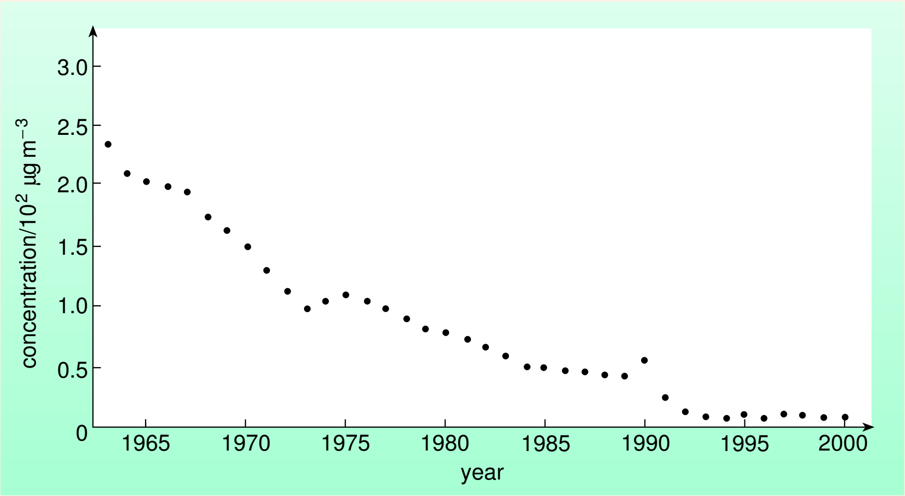


Figure 5.5: Average annual concentrations of sulphur dioxide in Tameside, Greater Manchester, 1963–2000.

Note how the vertical axis in Figure 5.5 is labelled. Concentrations have been expressed in micrograms per metre cubed ($\mu g\,m^{-3}$), so the quantity represented along the vertical axis has been *divided* by $\mu g\,m^{-3}$, in the same way as the temperature in Figure 5.4b was divided by °C. But in Figure 5.5, the quantity has been divided not only by its units but also by a power of ten. This can be a useful strategy in graph plotting because it allows manageable numbers to be used in labelling the divisions on the axis. To obtain the actual value of a quantity corresponding to a particular tick mark on the axis, we have to multiply the value given at the mark by the power of ten and by the units. For example, the mark labelled 1.5 represents:

$$1.5 \times 10^2\ \mu\text{g m}^{-3} = 150\ \mu\text{g m}^{-3}$$

Another way of looking at this is to say that a measured concentration has first been expressed in scientific notation:

$$150\ \mu\text{g m}^{-3} = 1.5 \times 10^2\ \mu\text{g m}^{-3}$$

and then reduced to a pure number by dividing it by the power of ten and the units:

$$\frac{1.5 \times \cancel{10^2}\ \cancel{\mu\text{g}}\,\cancel{\text{m}^{-3}}}{\cancel{10^2}\ \cancel{\mu\text{g}}\,\cancel{\text{m}^{-3}}} = 1.5$$

Figure 5.5 gives a clear visual image of a downward trend in sulphur dioxide concentration, but occasional blips such as occurred in 1990 mean that it is still not possible to use earlier data to predict future concentrations with any certainty. There are simply too many variables that can affect the concentration of atmospheric sulphur dioxide. In other circumstances, for instance when the two quantities plotted are linked by an equation, it *is* possible to use a graph for predictive purposes.

As an example of linked quantities, consider the data in Table 5.1 relating the mass of a series of aluminium spheres to their diameter. The data are plotted in Figure 5.6. Notice that the columns of the table have been labelled according to the same convention used to label the axes of the graph.

Diameter/10^{-2} m	Mass/10^{-3} kg
0.4	0.1
0.5	0.2
0.7	0.5
1.0	1.4
1.3	3.1
1.5	4.8
1.8	8.2
2.0	11.4

Table 5.1: Masses of aluminium spheres of different diameters

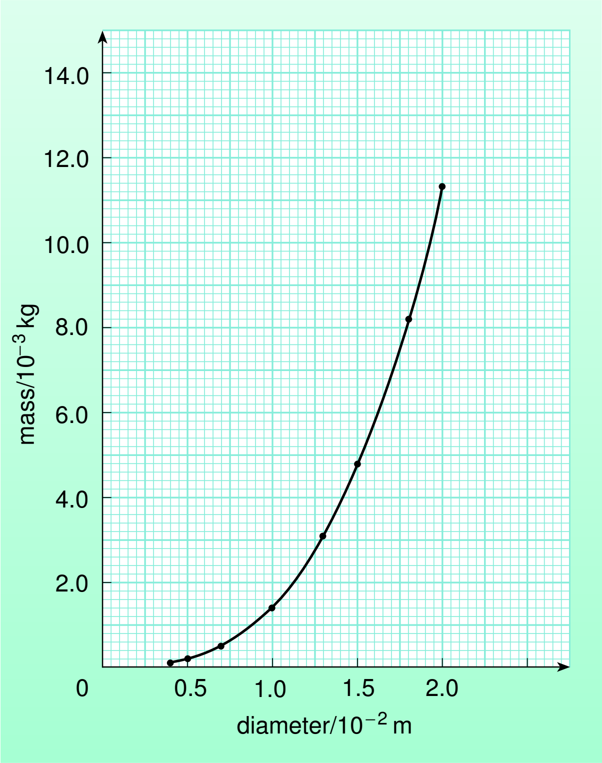


Figure 5.6: Graph of the masses of aluminium spheres of different diameters.

Question

What is the diameter in centimetres of the smallest sphere?

Answer

The diameter of the smallest sphere is obtained by multiplying 0.4 by the power of ten and the units:

diameter $= 0.4 \times 10^{-2}$ m $= 0.4$ cm

In fact, the mass M of a sphere of diameter d, made of material of density ρ, is given by the equation $M = \pi\rho d^3/6$. The data have been calculated and the graph constructed using this formula, so all the points lie on a smooth curve. When the axes of a graph represent quantities that are connected by an equation, the data points should never be joined in the jagged point-to-point way used in Figure 5.3 and Figure 5.4. Instead, a smooth line should be drawn through them. As you will see later in this chapter, a line described as 'smooth' may be straight, or may be curved in any direction, or may have humps and dips. Smoothness depends on the absence of abrupt changes in direction, not on shape.

Once the line has been drawn on Figure 5.6, we can use the graph to find intermediate values. This graph has been drawn on graph paper to make it easier to read values from it. You should start by working out the scale used on each axis. On this occasion tick marks have been drawn every 0.5×10^{-2} m on the horizontal axis, so each feint grid line represents 0.05×10^{-2} m; on the vertical axis the tick marks are every 2.0×10^{-3} kg so each feint grid line represents 0.2×10^{-3} kg.

Question

What would be the mass of an aluminium sphere of diameter 1.6 cm?

Answer

To find the mass corresponding to a diameter of 1.6 cm (i.e. 1.6×10^{-2} m) we need to find the point on the horizontal axis representing this diameter and draw a line vertically upwards from there until it meets the curve. We then draw a line horizontally from that intersection to meet the vertical axis and read off the corresponding mass. Print Figure 5.6 and draw these lines directly on to it using the grid lines on the graph paper to help you. You should find that the mass corresponding to a diameter of 1.6 cm is 5.8×10^{-3} kg (i.e. 5.8 g).

This process of reading *between* points plotted on a graph, in order to find corresponding intermediate values of the plotted quantities, is called interpolation.

Provided we are sure that the equation connecting the two quantities is valid even outside the plotted range, we can also extend the line on the graph to determine corresponding values of the quantities that are larger or smaller than those plotted.

Question

What would be the mass of an aluminium sphere of diameter 2.1 cm?

Answer

To find the mass corresponding to a diameter of 2.1 cm (i.e. 2.1×10^{-2} m) we need to find the point on the horizontal axis representing this diameter and draw a line vertically upwards from there. Then (and this is the difficult bit!) we have to extend the curve until it meets this vertical line. We then draw a line horizontally from that intersection to meet the vertical axis and read off the corresponding mass. Print Figure 5.6 and try drawing the lines. If your drawing skills are high, you should obtain a mass of 13.1×10^{-3} kg, but most people find it extremely difficult to draw smooth curves freehand, so if you obtain a value between 12.8×10^{-3} kg and 13.4×10^{-3} kg you have done well.

This process of extending a graph beyond the highest or lowest data points, in order to find corresponding values of the plotted quantities outside the original range, is called extrapolation. Extrapolation is always particularly difficult in regions where graphs curve, or have very steep or very shallow slopes. The latter situation applies to Figure 5.6 in the region where the diameter becomes very small. It would be practically impossible to determine by extrapolation the mass corresponding to a diameter of, say, 0.2 cm. All we can legitimately say is that if the diameter is zero, the mass will also be zero, so the curve must go through the point at which the axes meet. On any graph the point at which both plotted quantities are equal to zero is called the origin.

The fact that the graph in Figure 5.6 is curved makes both interpolation and extrapolation more uncertain than they would be if the graph was a straight line. In Question 5.1 you can practice these processes using a graph that is easier to deal with.

Question 5.1 Answer

Five measurements have been made to investigate the way in which the voltage across the terminals of a power supply varies according to the current flowing in the circuit. The data are plotted on Figure 5.7. (The SI unit of voltage is the volt, symbol V; the SI unit of electric current is the ampere, symbol A.)

(a) What is the value of the voltage when the current is 1.5 A?

(b) What is the value of the voltage when the current is zero?

(c) What is the value of the current when the voltage is zero?

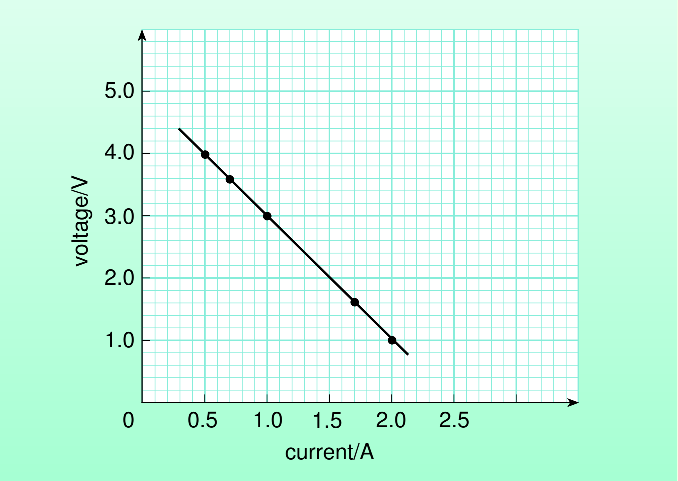


Figure 5.7: Measurements of voltage against current for the circuit in Question 5.1.

5.2 Straight-line graphs

As you have seen, it is possible to obtain useful insights and information from curved graphs such as the one in Figure 5.6, and we will return to the interpretation of curved graphs in Section 5.4. However, if data can be presented in the form of a straight-line graph, the analysis becomes more straightforward. As you will have discovered for yourself by doing Question 5.1, if you need to determine the values of quantities lying between those that were actually measured, it is slightly easier to perform the interpolation on a straight line than on a curve. And if you need to estimate values of quantities outside the original range of measurements, it is *considerably* easier to extrapolate a straight line than a curve. Furthermore, it is often possible to use a straight-line graph to obtain additional quantities, other than those measured. For example, the range of speeds at which the Earth's tectonic plates move was given in Box 3.1, but it is not possible to make a direct measurement of these speeds. Scientists have to deduce them by measuring other quantities and plotting graphs of their results.

5.2.1 The gradient of a straight-line graph

Box 5.3 gives a brief outline of the phenomenon of sea-floor spreading, the action of which is to split the Earth's surface and move sections of the crust apart. In order to work out the rate at which the separation takes place, Earth scientists date the rocks and measure the separation of rocks of the same age.

Table 5.2 shows some typical data. (Remember from Section 2.2 that Ma is the abbreviation for 'million years'.)

As discussed in Section 5.1.2, labelling the left-hand column as 'Age of rock/Ma', and the right-hand column as 'Separation distance/km', means that pure numbers can be entered in each row of the table.

Age of rock/Ma	Separation distance/km
0.78	17
0.99	18
1.07	21
1.79	32
1.95	39
2.60	48
3.04	58
3.11	59
3.22	62
3.33	65
3.58	68

Table 5.2: Positions of some dated areas either side of the mid-Atlantic ridge south-west of Iceland

Figure 5.9 shows a graphical representation of the data from Table 5.2. Although it is obvious just from the table that the separation distance increases with age, the graph immediately gives more information. First, it tells us about the relationship between the quantities plotted: the points lie pretty much on a straight line. The relationship between the age and the distance is thus said to be linear. Secondly, the graph provides a good test of the reliability of the data. It is clear that there are no 'rogue points' lying well off the straight line. However, the points do not all lie *exactly* on a single line. The black line that has been drawn through them is the 'best-fit line' — i.e. the line that is most representative of the data as a whole. Best-fit lines usually only go through some of the data points (and need not necessarily go through any); there should be approximately the same number of points above and below the line. The line has also been drawn to go through the origin, the point at which age is 0 Ma and distance is 0 km. This has been done because it is clear that newly-formed crust will not have moved any distance.

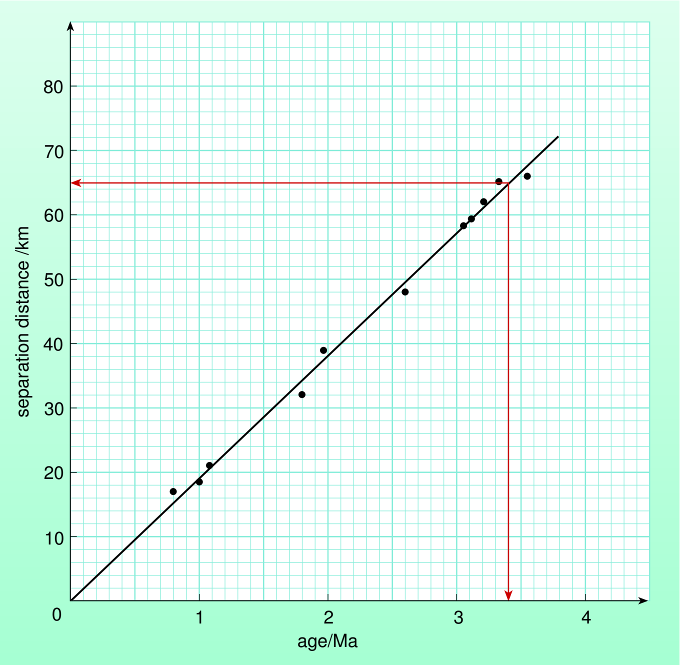


Figure 5.9: Graph of data in Table 5.2. The black line represents the 'best-fit' to the data. The red lines show that ocean crust of age 3.4 Ma has separated by 65 km.

The aim of collecting the data for age and separation distance was to calculate the *rate* of sea-floor spreading and this calculation can be made directly from the graph. For an object moving at a steady rate, the speed v is related to the distance d travelled in a time t by the equation:

$$v = d/t$$

The red lines on Figure 5.9 show that, according to the best-fit to the data, ocean crust of age 3.4 Ma has separated by 65 km. So the average spreading rate is:

$$v_{av} = 65\ \text{km}/3.4\ \text{Ma} = 19\ \text{km}\,\text{Ma}^{-1}\ \text{(to 2 significant figures)}$$

Now you could of course carry out similar calculations using any of the individual data pairs in Table 5.2. For example from the first data pair:

$$v_1 = 17\ \text{km}/0.78\ \text{Ma} = 22\ \text{km}\,\text{Ma}^{-1}$$

and from the fourth pair:

$$v_4 = 32\ \text{km}/1.79\ \text{Ma} = 18\ \text{km}\,\text{Ma}^{-1}$$

The first pair corresponds to a point that lies above the best-fit line and therefore gives a value of v that is higher than that calculated from the graph, while the point corresponding to the fourth data pair lies below the line and consequently gives a value of v that is lower than that calculated from the graph. If we wanted to calculate the average spreading rate directly from the tabulated data, the best we could do would be to calculate values from each of the eleven data pairs in the table (i.e. v_1 to v_{11}) and then average all these speeds. Plotting a graph therefore saves a tedious amount of calculation: using the best-fit line allows v_{av} to be calculated in a single step. In other words, a graph provides a reliable way of averaging results.

Question

What can you deduce from the fact that all the data points are close to the best-fit straight line, with some points lying above and others below the line?

Answer

The rate of spreading has remained roughly constant over time. Again the graph provides this information at a glance, whereas it would require a lot of calculation to deduce it from the raw data in Table 5.2.

Another way of describing this process of calculating the spreading rate from the distance–time graph is to say that we have determined the 'slope' or gradient of the best-fit line. Figure 5.10 shows the analogy with the gradients used to characterize steep hills, which you may have seen on road signs. The gradient is defined in this context as the 'rise' (the total change in vertical distance) divided by the 'run' (the total change in horizontal distance). It is important to remember that the actual distance travelled along the road is not involved in the calculation. Note that in this particular case the gradient has no units, because it is calculated by dividing a length by another length. In general, however, gradients must, as with the example of Figure 5.9, be given their correct units. In the case of a road, it is common to quote the gradient in the form of a percentage (33% in the case of Figure 5.10). With a graph it is more usual to quote the gradient as a single number.

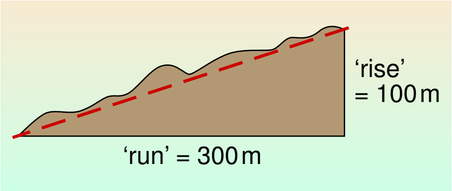


Figure 5.10: Vertical cross-section through a road. The gradient of this road is given by: 'rise'/'run' = 100 m/300 m,
i.e. gradient is 1/3 or 33%.

The gradient of a straight line is the same all the way along it, so any two points on the graph can be used to define the rise and the corresponding run. If, as is the case in Figure 5.11a, a graph goes through the origin, it may be convenient to use that fact in calculating the gradient; here the rise is $(y_2 - 0)$ and the run is $(x_2 - 0)$, so there are no subtractions to do. This was effectively the technique used in calculating the sea-floor spreading rate from Figure 5.9, when just one point on the best-fit line was chosen from which to calculate the speed.

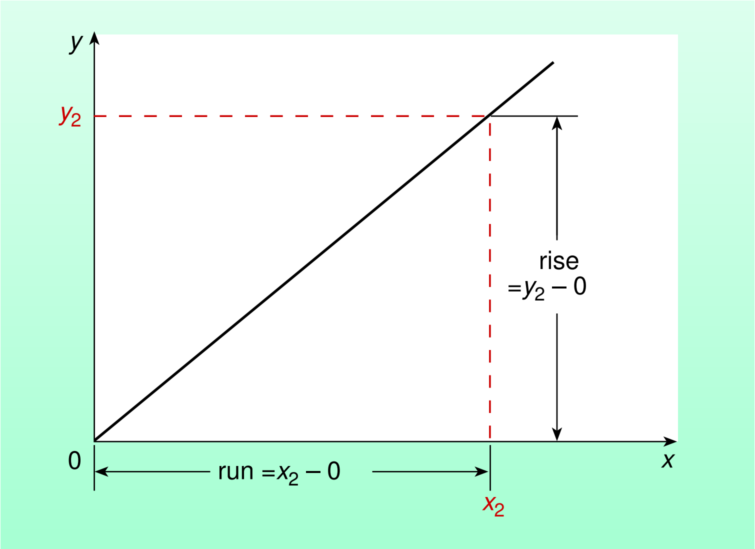


Figure 5.11a: This straight line goes through the origin, so its gradient $= \dfrac{y_2 - 0}{x_2 - 0} = \dfrac{y_2}{x_2}$

However, not all graphs go through the origin, so the method illustrated by Figure 5.11a is not always applicable. Figure 5.11b shows the most general method of determining the gradient of a straight-line graph, which can be used whether or not the line goes through the origin.

For a straight-line graph in which the value y_2 on the vertical axis corresponds to a value x_2 on the horizontal axis, and a value y_1 on the vertical axis corresponds to a value x_1 on the horizontal axis:

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

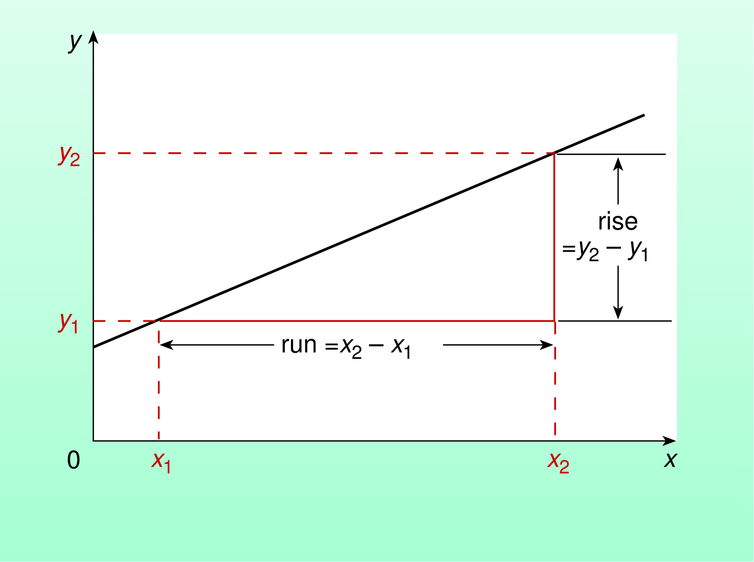


Figure 5.11b: For any straight line, the gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

Whatever points are chosen for determining the rise and run, it is always a good idea to choose ones that are easy to read on at least one axis and preferably on both axes! It is also good practice to choose points as widely separated as possible.

Worked example 5.1

When light is shone onto certain metals, electrons are emitted from the metal. This phenomenon is called the photoelectric effect, and will be described in more detail in Box 5.4. Figure 5.12a shows a graph arising from a photoelectric experiment on a particular metal, relating the energy of the ejected electrons to the frequency (i.e. the colour) of the light falling upon the metal. The energy is measured in joules (symbol J) and the frequency in units of s^{-1} (which are better known as hertz). What is the gradient of this graph?

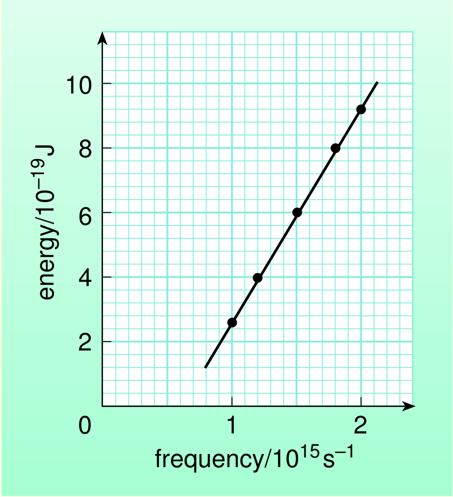


Figure 5.12a

Answer

It is clear that even if the line were to be extrapolated to smaller values of energy and frequency it would not go through the origin, so the method shown in Figure 5.11b is the appropriate one to use in calculating the gradient.

From the lines drawn on Figure 5.12b,

$$\text{gradient} = \frac{(9.2 \times 10^{-19}\ \text{J}) - (2.6 \times 10^{-19}\ \text{J})}{(2.0 \times 10^{15}\ \text{s}^{-1}) - (1.0 \times 10^{15}\ \text{s}^{-1})}$$
$$= \frac{(9.2 - 2.6) \times 10^{-19}\ \text{J}}{(2.0 - 1.0) \times 10^{15}\ \text{s}^{-1}}$$
$$= \frac{6.6 \times 10^{-19}\ \text{J}}{1.0 \times 10^{15}\ \text{s}^{-1}}$$
$$= 6.6 \times 10^{(-19-15)}\ \frac{\text{J}}{\text{s}^{-1}}$$
$$= 6.6 \times 10^{-34}\ \text{J s} \quad \left(\text{remembering that } \frac{1}{\text{s}^{-1}} = \text{s}\right)$$

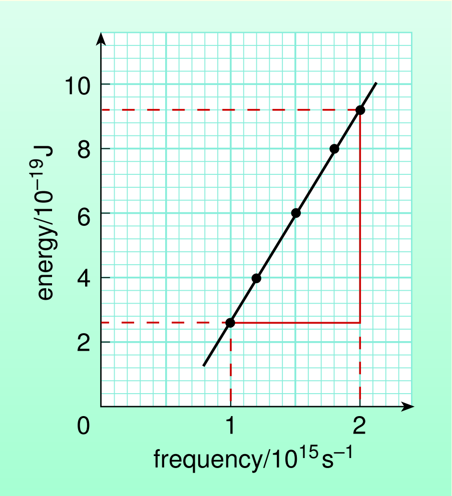


Figure 5.12b

Note that on this occasion the line drawn passes through, or very close to, all the data points. If the best-fit line *does not* go through all the data points, care must be taken to calculate the gradient of the graph from the line rather than from just two data points.

Question 5.2 Answer

The speed of seismic waves (see Box 3.1) may be calculated by measuring the time for the waves to reach measuring instruments at different distances from the epicentre of the earthquake. Some typical data from such a series of measurements on P waves are plotted in Figure 5.13. Use the graph to calculate the average speed of the P waves.

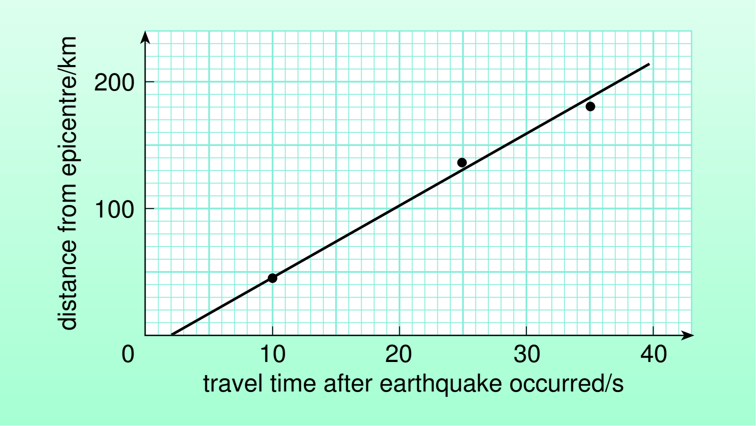


Figure 5.13: Graph showing how long it takes for P waves from a shallow-focus earthquake to reach three detectors at different distances from the epicentre. (Note that the focus is the point within the Earth at which the seismic event takes place, and the epicentre is the point on the Earth's surface vertically above the focus.)

5.2.2 Dependent and independent variables

In Figure 5.13, the time was deliberately plotted on the horizontal axis and the distance travelled on the vertical axis, so that the gradient would be equivalent to the seismic wave speed. However, for this particular example, plotting the graph this way round is not standard practice. The convention that scientists follow is to plot on the *horizontal* axis the variable that is under their control. Because they can choose the values of this quantity, it is called the independent variable. In the case of the measurements described in Question 5.2, there is a choice (within reason) of where the seismic wave detectors are located; therefore distance from the epicentre is the independent variable. The time taken for the P waves to arrive depends on where the detectors have been positioned, so this is called the dependent variable. According to the convention, the dependent variable is plotted on the *vertical* axis. Figure 5.14 shows the same data as Figure 5.13, but replotted so that the convention is followed. The three points correspond to those on Figure 5.13 and a best-fit line has again been drawn.

The seismic wave speed can be calculated equally well from Figure 5.14 as from Figure 5.13.

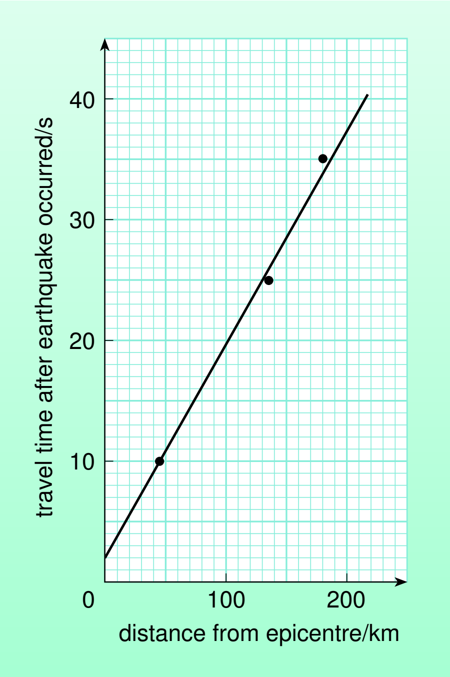


Figure 5.14

Question

What will be units of the gradient of the graph in Figure 5.14?

Answer

The gradient will have units of seconds divided by kilometres, which can be written either as s/km or as $\mathrm{s\,km^{-1}}$.

Question

In Question 5.2, you calculated the speed of the seismic wave in units of km/s (or $\mathrm{km\,s^{-1}}$). How are the units $\mathrm{s\,km^{-1}}$ related to these units of speed?

Answer

The units $\mathrm{s\,km^{-1}}$ and $\mathrm{km\,s^{-1}}$ are reciprocals: i.e. $\dfrac{1}{\mathrm{s\,km^{-1}}} = \mathrm{km\,s^{-1}}$.

Therefore to calculate the speed of the P waves from the time against distance graph of Figure 5.14, we need to determine the gradient and then take its reciprocal.

Question 5.3 Answer

Use Figure 5.14 to determine the average speed of the seismic waves. Remember to use the correct units at each stage of your calculation. Does your final answer agree with the value you obtained in Question 5.2?

5.2.3 Interpreting straight-line graphs and gradients

The graphs we have looked at so far in this section have all sloped up from left to right. But graphs can slope the other way too. Figure 5.15 shows the result of measuring the depth of snow in a particular location over a period of time, plotted on a graph of depth against time.

When describing a graph, the convention is to state the dependent variable first; a graph of 'depth against time' therefore plots depth on the vertical axis and time on the horizontal axis.

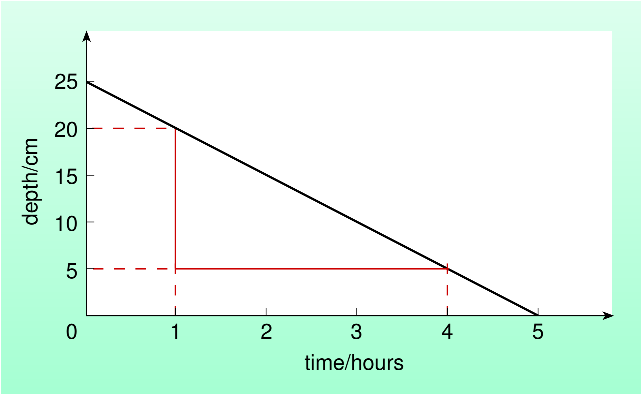


Figure 5.15: Depth of snow measured over a five-hour period.

For the line drawn in Figure 5.15, gradient is given, as before, by:

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

If x_1 is 1 hour and x_2 is 4 hours, the corresponding y values are $y_1 = 20$ cm and $y_2 = 5$ cm, i.e. x_2 is greater than x_1 but y_1 is greater than y_2. This means that:

$$\text{gradient} = \frac{(5-20)\text{ cm}}{(4-1)\text{ hours}} = \frac{-15\text{ cm}}{3\text{ hours}} = -5\text{ cm hour}^{-1}$$

In other words, the gradient is negative.

Question

What physical meaning do you attach to the gradient in this context?

Answer

The graph shows that depth is decreasing with time — in other words the snow is melting. The negative value of the gradient conveys this same information. The gradient is constant over the time during which the measurements have been made, so the snow is melting at a steady rate.

Now look at Figure 5.16, which shows the variation of distance from a given point with time, for four objects A to D moving in a variety of situations. A scientific way to say this is that the graphs all show distance 'as a function of' time, or d as a function of t.

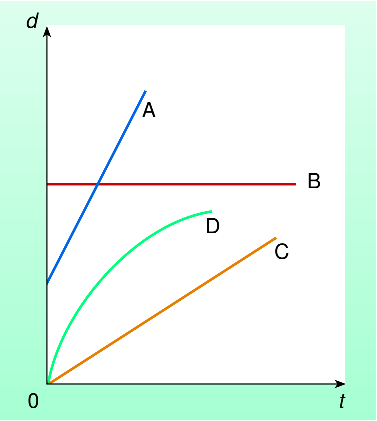


Figure 5.16

In general, it is the dependent variable (which by convention is plotted along the vertical axis) that is described as being a function of the independent variable (which is plotted along the horizontal axis).

So in the situations shown in Figure 5.16, time is the independent variable: the experimenter has chosen specific times at which to make the measurements and has recorded the position of the object at those times.

As with Figure 5.13, the gradient of each line gives the speed with which that particular object is moving.

Question

Which, if any, objects are moving with constant speed? Of these, which is travelling the most quickly?

Answer

The gradients of the distance–time graphs for objects A and C are constant, so their speed is constant. The gradient of the line for object A is greater (i.e. the distance–time graph is steeper) than that for object C, so A is moving at a higher speed than C.

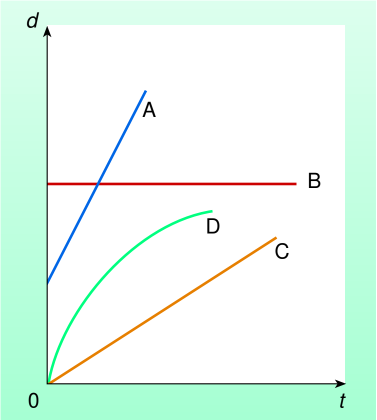


Figure 5.16

Question

What is happening to object B? What is the gradient of the line for object B on the graph?

Answer

For object B the distance travelled is not changing with time. The most likely explanation of this is that the object is stationary. On the graph for this object, the rise is always zero, so the gradient of the graph is also zero. This is simply another way of saying that its speed is zero.

Question

What is happening to object D?

Answer

The gradient for object D gradually decreases (i.e. gets less steep). In other words the object is slowing down.

Question 5.4 Answer

The lowest level of the Earth's atmosphere is called the troposphere. Figure 5.17 shows the variation in temperature of the troposphere from sea-level to an altitude of about 2.5 km. Estimate to two significant figures the gradient of this graph. (Because you are only being asked for an estimate, you do not need to attempt great precision in reading values off the graph, but you should be careful over signs and units.) Describe clearly, in one sentence, what your result means.

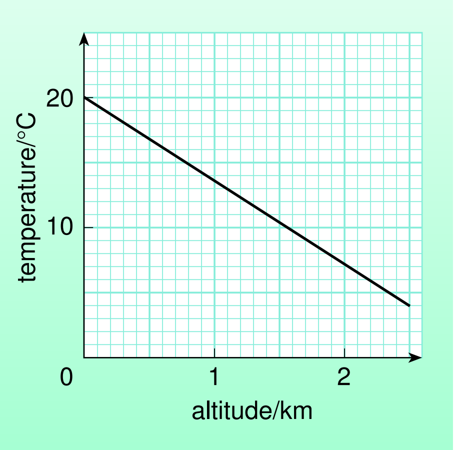


Figure 5.17

Question 5.5 Answer

At higher levels in the troposphere, the temperature drops still further. Figure 5.18 shows the variation in temperature for altitudes between 4 km and 11 km above sea-level. Estimate to two significant figures the gradient of this graph. Does your answer agree with that for Question 5.4?

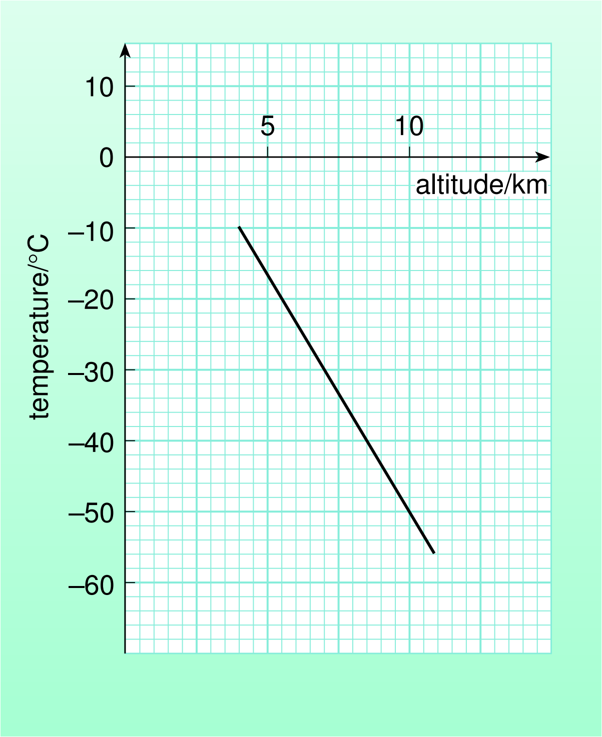


Figure 5.18

5.3 The equation of a straight line

In the preceding sections, you saw how useful information can be derived from a straight-line graph by interpolation, extrapolation or calculation of the gradient. But this does not exhaust the potential of a graph as a tool: it becomes even more useful when it can be matched to an equation.

5.3.1 Proportional quantities

Two quantities are said to be proportional to each other, or more precisely to be directly proportional to each other, if multiplying (or dividing) one by a certain amount automatically results in the value of the other being multiplied (or divided) by the same amount. If I buy 500 litres of heating oil I pay twice as much as if I had bought 250 litres, but one-half as much as if I had bought 1000 litres — assuming that on such amounts there is no bulk discount. The cost is directly proportional to the volume. We can write this succinctly in the form:

$$\text{total cost} \propto \text{volume}$$

where the symbol $\propto$ stands for 'proportional to'. To determine the total cost of something we multiply the number of items we are buying by the price per item, so we can turn our original proportionality relationship into an equation of the form:

$$\text{total cost} = (\text{cost per litre}) \times (\text{volume in litres})$$

We are assuming that the cost per litre is constant however big the delivery. This constant factor, which is required to turn the proportionality into an equation, is called the constant of proportionality.

Now consider how this relationship between cost and volume appears on a graph, such as that plotted in Figure 5.19. If I don't buy any oil, the cost is zero (but the heating doesn't work!), so the graph must go through the origin. If I buy 500 litres it costs £100, and 1000 litres cost £200.

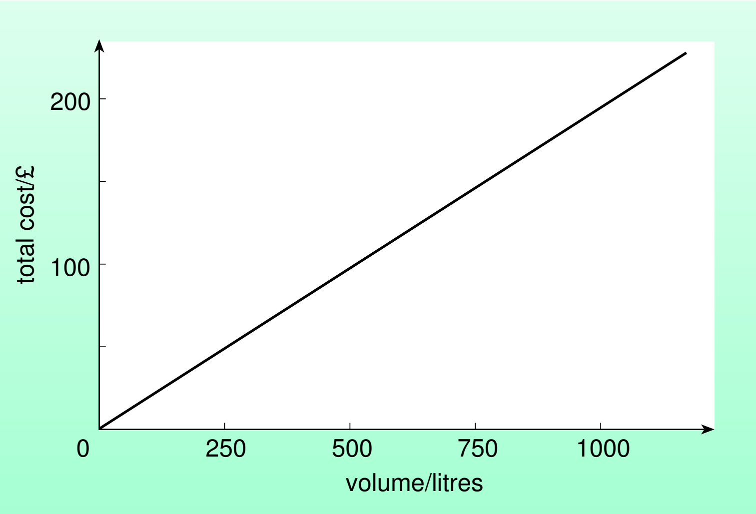


Figure 5.19: The cost of heating oil.

Question

What is the gradient of this graph? What does that value represent?

Answer

The gradient is $\dfrac{£(200-0)}{(1000-0)\text{ litre}} = \dfrac{£200}{1000\text{ litre}} = £0.20/\text{litre} = 20\text{ pence/litre}$

In other words the gradient represents the cost per litre. The gradient of the graph is the constant of proportionality between total cost and volume of oil.

Generalizing from this example:

if $y = kx$, where y and x are variables and k is a constant,

then y is said to be directly proportional to x,

i.e. $y \propto x$.

A graph of y against x will go through the origin and have gradient k, as illustrated in Figure 5.20.

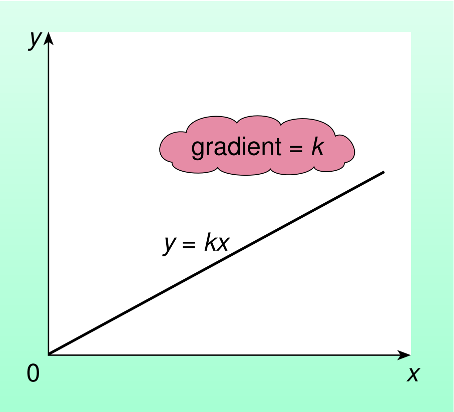


Figure 5.20: A graph of $y = kx$

Graphs like Figure 5.20, that by their shape show the nature of the relationship between quantities but do not have scales marked on the axes, are called 'sketch graphs'. They can be very useful for illustrating ideas, without the need for accurate plotting or drawing.

Question 5.6 Answer

Figure 5.21 shows the graphs corresponding to two different relationships between a variable v and another variable z. The quantities r and s are constants. Which is larger, r or s?

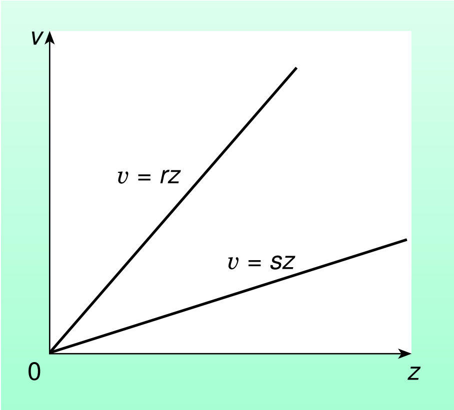


Figure 5.21: Two proportional relationships: $v = rz$ and $v = sz$.

Question 5.7 Answer

Figure 5.22 shows three sketch graphs. Which of them represents a relationship between directly proportional quantities?

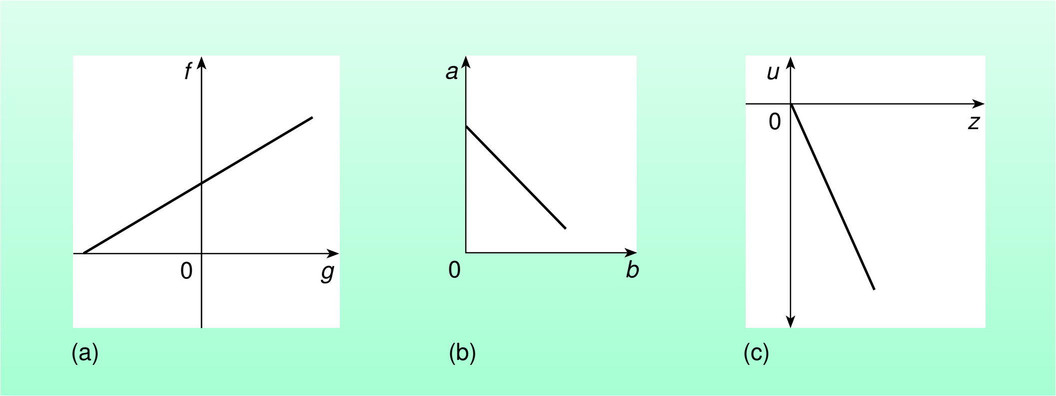


Figure 5.22: Sketch graphs for use with Question 5.7.

5.3.2 A general equation for a straight line

Returning to the example of the oil delivery, suppose a different company decided that it would sell at a lower cost per litre, but would impose a fixed delivery charge in addition to the price of the oil. This situation is represented by an equation of the form

$$\text{total cost} = (\text{cost per litre} \times \text{volume in litres}) + \text{delivery charge}$$

and this is plotted on the graph in Figure 5.23.

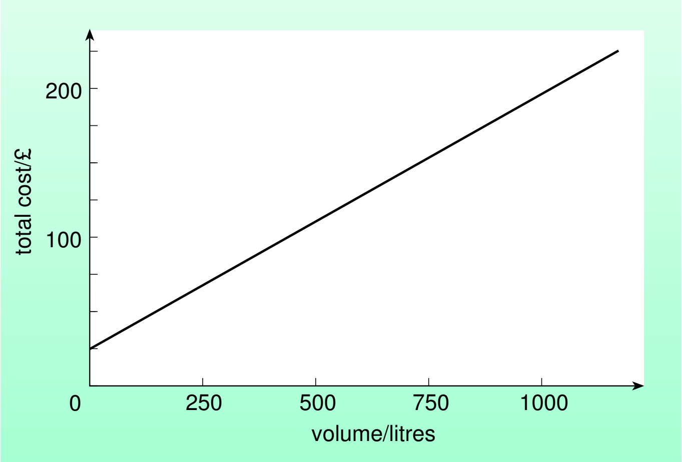


Figure 5.23: Graph of the cost of heating oil as a function of volume delivered.

Question

From Figure 5.23, estimate both the price per litre and the delivery charge.

Answer

The cost per litre is still given by the gradient of the graph, which is this case is approximately

$$\frac{£(200 - 25)}{(1000 - 0)\ \text{litre}} = \frac{£175}{1000\ \text{litre}} = 17.5\ \text{pence/litre}$$

The fixed charge can be estimated from the point at which the line crosses the vertical axis: at this point, there is no charge for oil (since the volume is zero) so the fixed charge represents the only contribution to the total cost. The delivery charge is therefore £25.

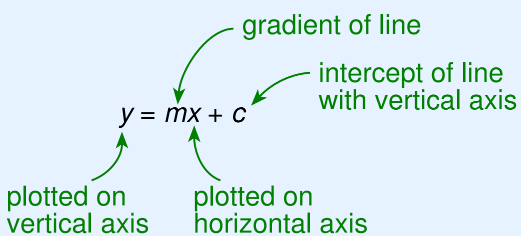
Note that what this company is effectively doing is giving a discount for bulk buying compared to the arrangement described by the graph of Figure 5.19. For a delivery of 1000 litres, the cost is identical whichever company is used. For less than 1000 litres, it would be cheaper to buy from the first company. For volumes larger than 1000 litres the second company offers the better deal.

Generalizing from this example, if two quantities y and x are related by an equation of form

$$y = mx + c \tag{5.1}$$

where m and c are constants, then a graph of y against x will be a straight line that does not go through the origin. The graph will have gradient m. And when $x = 0$, then $y = c$, so the graph crosses the vertical axis at c. The point at which a line on a graph crosses an axis is called the intercept of the line with that axis. This is illustrated in Figure 5.24.

The equation of a straight line is commonly written in the form



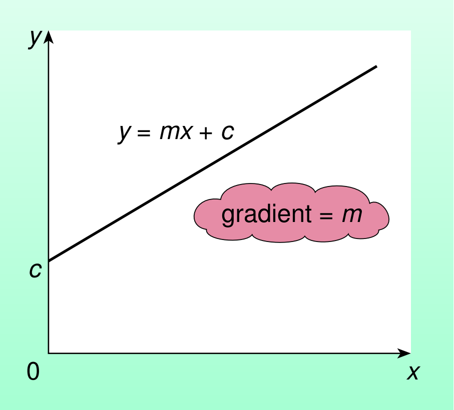


Figure 5.24: A straight-line graph with gradient m and intercept c on the vertical axis.

Although the general equation of a straight line is most usually written in the form $y = mx + c$, it is important to remember that the letters used and their order are quite arbitrary: $v = u + at$ is also the equation of a straight line. Also, although $y = mx + c$ does not contain any minus signs, both the gradient m and the constant c might have a negative value.

Question

If $v = u + at$ and v is plotted against t what, in terms of the symbols in the equation, are the values of the gradient of the graph and the intercept on the vertical axis?

Answer

$v = u + at$ can be rearranged as $v = at + u$. Comparison with the standard equation of a straight line

$$y = \overset{\text{gradient}}{m}\,x + \overset{\text{intercept}}{c}$$
$$v = a\,t + u$$

shows that the gradient of a plot of v against t is a and the intercept with the vertical axis is u.

An example of how the gradient and intercept of a straight line may be used to derive quantities of real interest to scientists is given in Box 5.4.

Box 5.4 Einstein's photoelectric equation

When light of particular colours is shone onto certain metals, electrons are emitted from the metal, as shown diagrammatically in Figure 5.25. Some of the energy of the light is used to remove the electrons from the metal; the amount of energy required to do this varies from metal to metal, and is called the 'work function' ϕ of the metal. Any energy left over is given to the escaping electrons:

$$\begin{pmatrix}\text{energy of}\\ \text{incident}\\ \text{light}\end{pmatrix} = \begin{pmatrix}\text{energy required to}\\ \text{remove electrons}\\ \text{from the metal}\end{pmatrix} + \begin{pmatrix}\text{energy of}\\ \text{ejected}\\ \text{electrons}\end{pmatrix}$$

This word equation can be rearranged as:

$$\begin{pmatrix}\text{energy of}\\ \text{ejected}\\ \text{electrons}\end{pmatrix} = \begin{pmatrix}\text{energy of}\\ \text{incident}\\ \text{light}\end{pmatrix} - \begin{pmatrix}\text{energy required to}\\ \text{remove electrons}\\ \text{from the metal}\end{pmatrix}$$

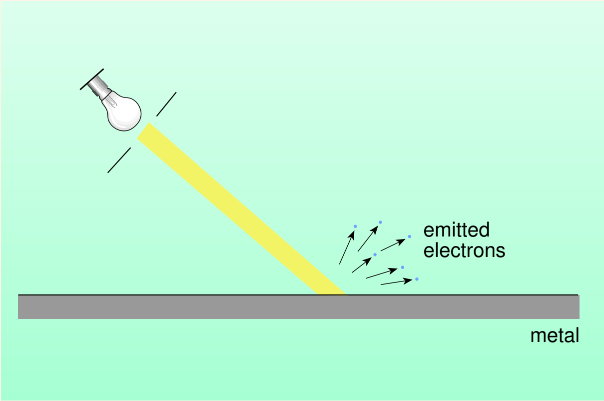


Figure 5.25: The photoelectric effect.

The colour of light is characterized by a quantity called its frequency f, and the energy of the incoming light is then given by hf, where h is a constant called Planck's constant. So the word equation above can be rewritten in the form:

$$E = hf - \phi$$

where the work function ϕ is a positive constant for any given metal. You saw in Figure 5.12 a typical graph of energy, E, against frequency, f. Comparison with the standard equation for a straight line shows how such a graph could be used to determine both h and ϕ. (Notice that the photoelectric equation contains a minus sign and therefore has to be slightly rearranged to allow direct comparison.)

$$
\begin{array}{cccccc}
y & = & m & x & + & c \\
E & = & h & f & + & (-\phi) \\
 & & \text{gradient} & & & \text{intercept}
\end{array}
$$

The gradient calculated in Worked example 5.1 (i.e. 6.6×10^{-34} J s) is therefore the value of Planck's constant h, and extrapolation of the line in Figure 5.12 to its intersection with the vertical axis could be used to determine the work function of the metal.

5.4 Graphs of different shapes

The previous section showed that it is a relatively straightforward matter to deduce the equation linking two variables when their relationship can be represented by a straight-line graph. But of course not all the quantities of interest in science are linearly related to one another. Suppose you were to plot one variable against another and obtained not a straight line but a curve. How could you then determine the relationship between the variables?

Imagine for example that you had taken a set of circular objects with radii 1, 2, 3, . . . 6 cm and measured their respective areas. Had you plotted the area A as a function of radius r you would have obtained a graph like that in Figure 5.26.

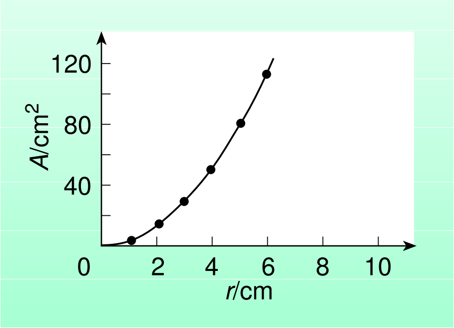


Figure 5.26: Area A of circles plotted as a function of their radii r.

Question

What is the equation relating the area A of a circle to its radius r?

Answer

$A = \pi r^2$

This equation shows that A is *not* directly proportional to r, so you should not have been surprised that plotting A against r did not give a straight line. In fact, the curved shape of Figure 5.26 is characteristic of a relationship involving the square of one of the quantities plotted. This particular shape is called a parabola.

Because in this case we know the equation relating A and r, it is quite easy to see how the curve of Figure 5.26 can be 'transformed' into a straight-line graph. A is equal to r^2 multiplied by a constant π. So although A is not directly proportional to r, it *is* directly proportional to r^2:

$$A \propto r^2$$

Therefore the result of plotting A against r^2 is a straight line, as illustrated in Figure 5.27.

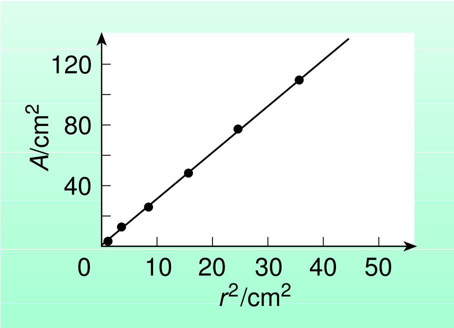


Figure 5.27: Areas A of circles plotted as a function of the squares of their radii r^2

Question

Without measuring anything on the graph itself, can you state the value of the gradient of the line in Figure 5.27?

Answer

Comparison with the standard equation for a straight line shows that

$$\begin{array}{cccccc} y & = & m & x & + & c \\ A & = & \pi & r^2 & (+ & 0) \end{array}$$

so the gradient of the line is π.

Worked example 5.2

The relationship between the distance s travelled by an object which has been dropped from a height and the time t for which it has been falling is

$$s = \tfrac{1}{2}gt^2$$

where g is a constant (the magnitude of the acceleration due to gravity). If you had measured the time as the object passed various points on the way down, how would you use a graph to determine the value of g from your data?

Answer

Since s is directly proportional to t^2, these are the variables to plot. The description of the experiment shows that s is the independent variable, which according to convention should be plotted on the horizontal axis.

We could rearrange the equation $s = \tfrac{1}{2}gt^2$ to give

$$t^2 = \frac{2s}{g}$$

So if t^2 is plotted against s, comparison with the standard equation for a straight line shows that

$$\begin{array}{ccccccc} y & = & m & x & + & c & \\ t^2 & = & \frac{2}{g} & s & (+ & 0 &) \end{array}$$

so the gradient of the line $= \dfrac{2}{g}$ and

$$g = \frac{2}{\text{gradient}}$$

{If you chose to plot s against t^2, then the gradient would be $\dfrac{g}{2}$, in which case $g = 2 \times \text{gradient}$.}

Question 5.8 Answer

Table 5.1 showed the mass of a number of aluminium spheres as a function of their diameters. When mass was plotted as a function of diameter, a curved graph (Figure 5.6) was obtained. The mass M of a sphere of diameter d, made of material of density ρ, is given by the equation $M = \pi\rho d^3/6$.

What quantities would you plot in order to obtain a straight-line graph from the data in Table 5.1?

What expression would be given by the gradient of the line?

Question 5.9 Answer

If you have ever regulated a long-case (grandfather) clock, you will know that the length of the pendulum, L, determines the period T (the time for one complete swing) and hence affects the accuracy with which the clock keeps time. For a simple pendulum, the period is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where, as in Worked example 5.2, g is a constant (the magnitude of the acceleration due to gravity). If you had measured T for various values of L, how would you use a graph to determine the value of g from your data?

{*Hint*: you may find it helpful to manipulate the equation so as to get rid of the square root.}

The trick of plotting quantities in such a way as to obtain a straight line is very useful when you want to discover the relationship between experimentally measured quantities. With practice, one can come to recognize curved graphs of various shapes, and this helps considerably in deciding how to transform the original data so as to obtain a straight-line plot. For example, if the result of plotting one quantity against another is a parabolic curve, this is an immediate indication that one of those quantities is proportional to the square of the other.

The rest of this section will simply introduce you to a few curves of different shapes and the equations to which they correspond. (An explanation of the techniques by which one can most easily take scientific data and discover what powers of the variables should be used in order to get a linear plot will come in Chapter 7.)

A completely different sort of curve is generated from experiments using the apparatus in Figure 5.28. This piston arrangement is designed for the study of a sample of gas. A pressure P can be applied to the piston and as the pressure increases so the volume V of the gas in the chamber will decrease. Conversely, if the pressure is reduced, the gas in the chamber will expand. If you have ever pumped up a bicycle tyre, you have probably noticed that when a gas is compressed it heats up, so in order to be sure that pressure and volume are the only variables involved in this particular experiment, it is important to ensure that each time the pressure is changed the gas is allowed to return to its original temperature T before the volume is measured. This temperature is maintained by the heat bath.

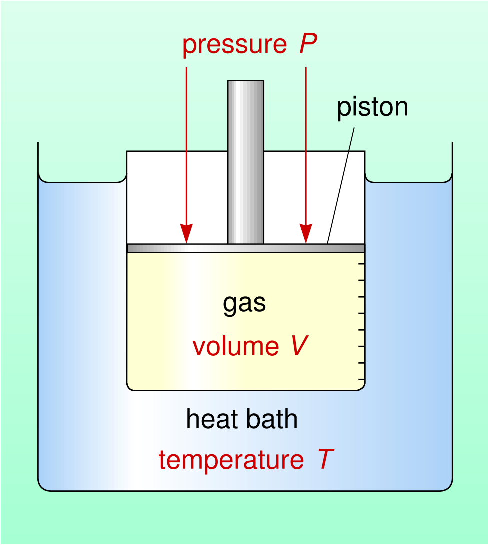


Figure 5.28: An apparatus for measuring how the volume of a sample of gas varies with the pressure at constant temperature.

A sketch graph showing the shape of a plot of V against P resulting from such an experiment is shown in Figure 5.29. A plot of this shape is called a hyperbola. A characteristic feature of the hyperbola is that as the variable on one axis approaches zero, the curve approaches more and more closely to the other axis but never actually touches it.

A hyperbola arises from plotting two quantities that are linked by one being directly proportional to the reciprocal of the other. In this case,

$$V \propto \frac{1}{P}$$

This could also be expressed in words by saying that 'V is directly proportional to one over P' but it is more usual to say that V is inversely proportional to P.

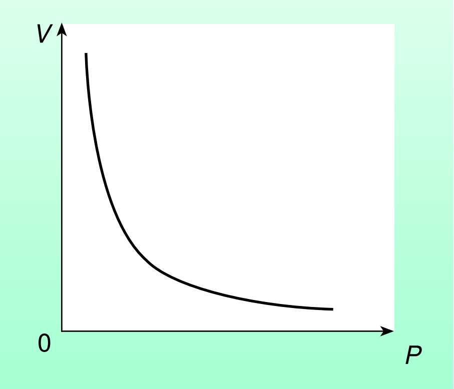


Figure 5.29: A graph of volume as a function of pressure for a fixed amount of gas at constant temperature.

In order to obtain a straight-line plot, we would therefore have to plot V against $1/P$, as illustrated by the sketch graph in Figure 5.30. In practice the volume of a real gas can never fall to zero, but if the line were extrapolated it would go through the origin.

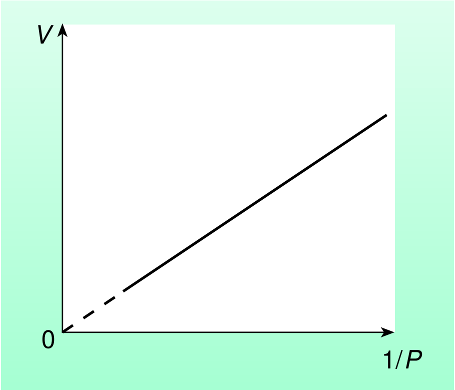


Figure 5.30: At constant temperature, the volume of a fixed amount of gas is inversely proportional to the pressure.

Question

If you wanted to plot a graph using measured values of P and V, but with P as the dependent variable, what would you have to plot on the horizontal axis in order to obtain a straight line?

Answer

We know that $V \propto \dfrac{1}{P}$, i.e. that $V = \dfrac{k}{P}$, where k is a constant.

Rearrangement gives $P = \dfrac{k}{V}$, i.e. $P \propto \dfrac{1}{V}$.

So a graph of P against $1/V$ would also be a straight line, as illustrated in Figure 5.31.

Note that graphs of $1/V$ against P and $1/P$ against V would also be straight lines.

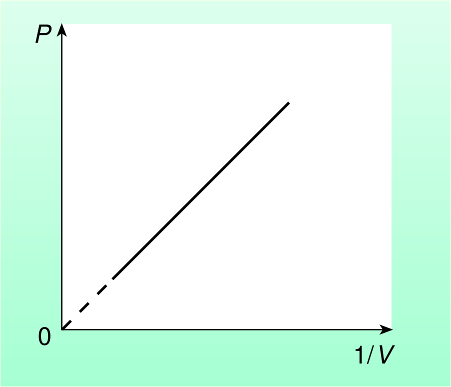


Figure 5.31: At constant temperature, the pressure of a fixed amount of gas is inversely proportional to the volume.

So far we have been primarily interested in the relationship between the pressure and the volume of the sample of gas, so the sketch graphs of Figures 5.29 to 5.31 correspond to a situation in which the temperature has been held constant. However, it would be equally possible to use the apparatus illustrated in Figure 5.28 to measure the volume of the gas sample as a function of temperature. Such measurements are the basis of the SI (kelvin) scale of temperature, which is discussed in Box 5.5.

Box 5.5 The absolute zero of temperature

Figure 5.31 shows that the pressure and volume of a fixed amount of gas at constant temperature are related by an equation of form $P = k/V$ where k is a constant, i.e.

$$PV = k \text{ (at constant temperature).}$$

This equation is a particular case of a more general equation which was introduced in Box 3.4, namely

$$PV = nRT \qquad (3.12)$$

where n is the number of moles of gas, R is the so-called gas constant, and T is the temperature (measured in kelvin).

Equation 3.12 can be rearranged to give

$$V = \frac{nR}{P}T$$

and if P is held constant then

$$\text{volume} = C \times \text{temperature}$$

where C is a constant equal to nR/P.

The value of C will depend on the value of P chosen, so if the volume of the sample of gas is measured as a function of temperature in three separate experiments, each one at a different constant pressure, three separate straight-line plots will be obtained, each with a different gradient. The larger the value of P chosen, the smaller the gradient will be.

Figure 5.32a shows how the volume of the sample of gas measured at three different pressures, varies over the temperature range 0 °C to 100 °C (note that the temperatures here are given in degrees Celsius). The really interesting aspect of the graph is that if the lines are extended to lower and lower temperatures, as shown in Figure 5.32b, they all meet at the same point on the horizontal axis, corresponding to a temperature of −273.15 °C, and to a volume of zero. Extrapolation beyond this point would imply a negative volume, which is impossible, so −273.15 °C is the lowest possible temperature. It is therefore known as the *absolute zero* of temperature. The SI (kelvin) scale of temperature sets this lowest possible temperature at 0 K.

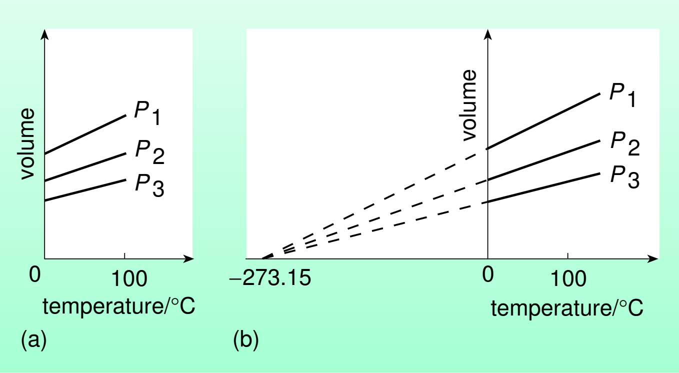


Figure 5.32: (a) At constant pressure, the volume of a fixed amount of gas is clearly related to the temperature. Here $P_1 < P_2 < P_3$. (b) Extrapolation shows that when the volume is zero then the temperature is −273.15 °C.

Temperatures may be converted from degrees Celsius to kelvin and vice versa using the word equation:

$$\begin{pmatrix}\text{temperature}\\ \text{in kelvin}\end{pmatrix} = \begin{pmatrix}\text{temperature}\\ \text{in degrees}\\ \text{Celsius}\end{pmatrix} + 273.15$$

Figure 5.32b is a reminder that at −273.15 °C (i.e. at 0 K) the volume is zero.

When T is expressed in kelvin, V is directly proportional to T, so the lines in Figure 5.33 go through the origin.

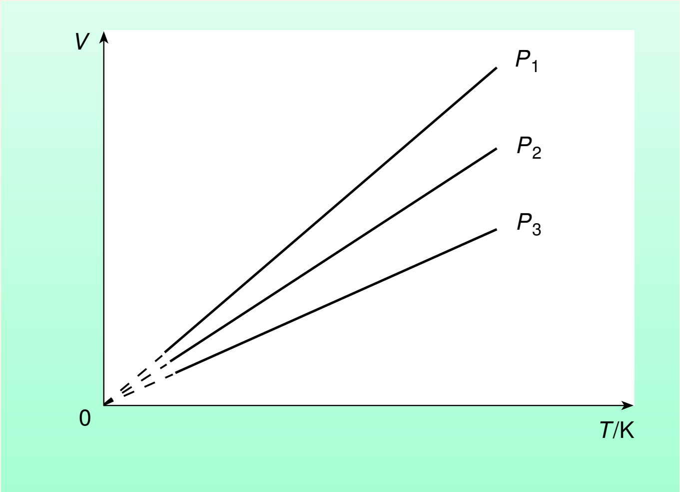


Figure 5.33: If T is measured on the kelvin scale, then $V \propto T$.

Yet another type of curved graph is obtained when the activity of a radioactive sample is monitored over time. The atoms of radioactive elements 'decay' by emitting small particles from their nuclei, thereby transforming themselves into atoms of quite different elements. These other elements may themselves be radioactive, or they may be stable. Radioactive decay is a random process, in that, although the total activity of a sample is predictable, one can never predict which individual nuclei are going to decay at any particular time. One form of polonium, the element named after the Polish homeland of Marie Curie, decays to leave stable atoms of lead. The activity of a sample of polonium is plotted as a function of time in Figure 5.34; the unit of activity is the bequerel (Bq), equal to 1 disintegration per second. Because disintegration of a polonium nucleus produces a stable lead nucleus, the number of unstable nuclei in the sample — and hence the activity — falls as time goes on.

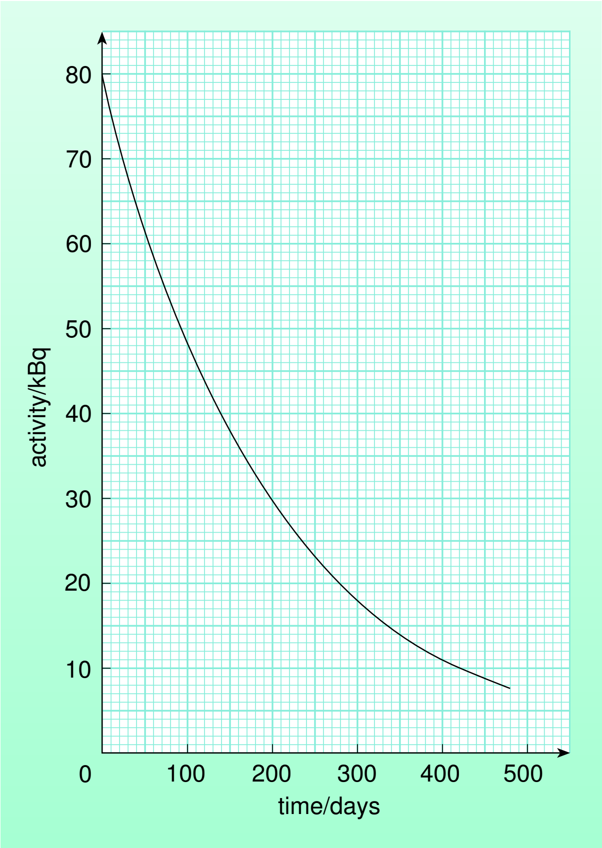


Figure 5.34: The activity of a sample of polonium as a function of time.

Question

How long does it take for the activity of the polonium sample in Figure 5.34 to drop to

(a) 40 kBq

(b) 20 kBq

(c) 10 kBq

Answer

Reading from Figure 5.34,

(a) the activity has dropped to 40 kBq after 140 days

(b) the activity has dropped to 20 kBq after 280 days

(c) the activity has dropped to 10 kBq after 420 days

A little further analysis shows that the time taken for the activity to drop:

- from 80 kBq to 40 kBq = 140 days
- from 40 kBq to 20 kBq = (280 – 140) days = 140 days
- from 20 kBq to 10 kBq = (420 – 280) days = 140 days

This result demonstrates a very important property of the curve plotted in Figure 5.34; whatever value of the quantity plotted on the vertical axis is chosen, the time taken for the quantity to fall to exactly one-half that value is a constant. This constant interval of time is known as the half-life, and curves that display this property are called 'exponential decays'. To the precision to which it is possible to read Figure 5.34, the half-life of the polonium sample is 140 days.

In radioactive decay, the activity is dependent on the number of radioactive nuclei present, which is usually denoted by the letter N. Figure 5.35 shows that if N_0 radioactive nuclei are present when timing starts (i.e. at time $t = 0$), then

- after one half-life $N = N_0 \times \frac{1}{2}$
- after two half-lives $N = \left(N_0 \times \frac{1}{2}\right) \times \frac{1}{2} = N_0 \left(\frac{1}{2}\right)^2$
- after three half-lives $N = N_0 \left(\frac{1}{2}\right)^2 \times \frac{1}{2} = N_0 \left(\frac{1}{2}\right)^3$
- so after n half-lives $N = N_0 \left(\frac{1}{2}\right)^n$

After a long time, and many half-lives, N will approach, though it will never reach, zero.

The equation describing the exponential decay shown in Figure 5.35 involves a special number, e. Like π and $\sqrt{2}$, e is an irrational number and to four significant figures its value is 2.718. The equation describing Figure 5.35 is

$$N = N_0\,\mathrm{e}^{-\lambda t} \tag{5.2}$$

where λ is a positive constant. In Chapter 7, you will discover the relationship between λ and the half-life for the decay, $t_{1/2}$. Then in Chapter 10 you will find that exponentials have another characteristic and defining property.

Question 5.10 Answer

Radium has a half-life of 1600 years. How long will it be before the number of radioactive atoms in a sample is reduced to $\frac{1}{16}$ of the number there are today?

Box 5.6 Dating meteorites and Moon rock

The age of many different natural materials can be determined from their radioactivity. Potassium is one element that is used to date rocks; potassium-40 has a half-life of 1.3×10^9 years, and decays to leave argon, an inert gas that does not combine with other elements. When rocks first form, they are molten, so any argon they might contain would simply escape into space. However, once the rocks solidify, any argon resulting from the radioactive decay of potassium-40 remains trapped. Geochemists can analyse the composition of a rock to determine the ratio of potassium to argon, and hence estimate a rock's age.

Dating using potassium and other radioactive elements has shown that almost all known meteorites are 4.6×10^3 Ma old, so their formation was contemporaneous with the formation of the Solar System. The oldest known Moon rock is 4.48×10^3 Ma old.

It is sometimes reported in the media that something is exhibiting 'exponential growth'. In fact, true exponential growth, in which the quantity being measured is multiplied by a constant factor over a given period of time, is a rather unusual phenomenon although it does occur. A general equation for exponential growth, analogous to Equation 5.2 for exponential decay, is

$$n = n_0\,\mathrm{e}^{at} \qquad (5.3)$$

where n_0 is the starting value of the quantity, n is its value after time t and a is a positive constant. Exponential growth is sometimes used as a model by biologists interested in the populations of organisms. Figure 5.36 illustrates the theoretical increase of yeast cells according to such a model. In practice, the death of organisms, as well as the influence of factors relating to overcrowding, will also affect the population, so that the increase in the number of organisms will not lie on a true exponential growth curve.

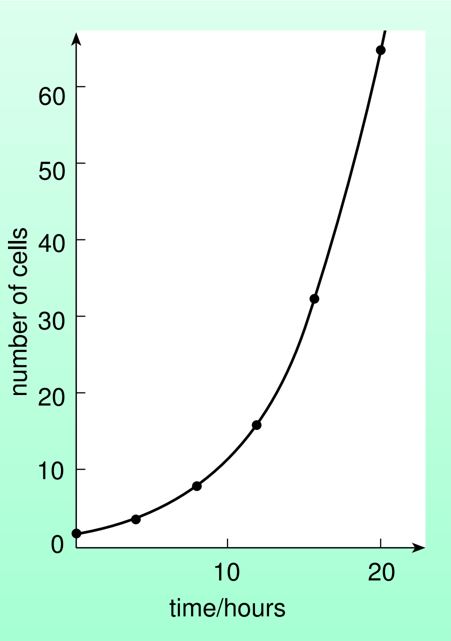


Figure 5.36: Model for the growth of yeast cells. The population consists of just two cells at time $t = 0$, and it is assumed that once in every four-hour period each cell divides into two cells, (i.e. the multiplying factor is 2).

5.5 Learning outcomes for Chapter 5

After completing your work on this chapter you should be able to:

5.1 demonstrate understanding of the terms emboldened in the text;

5.2 correctly interpret conventional labelling on graph axes or table columns, so as to deduce the power of ten and the units associated with a plotted or tabulated quantity;

5.3 use the processes of interpolation and extrapolation to read values from a graph;

5.4 calculate the gradient of a straight-line graph;

5.5 deduce the gradient and intercept of a straight-line graph from the equation of the line, and vice versa;

5.6 draw and interpret sketch graphs;

5.7 given the equation involving quantities raised to a power, decide what variable should be plotted in order to obtain a straight-line graph.

6 Angles and trigonometry

It is relatively easy to measure the distance along the ground from an observer to an object such as a tree, but measuring the height of the tree itself is rather less straightforward. Similarly, it is possible to find the distance from the Earth to the Moon by measuring the time taken for a laser beam to travel to the Moon and back, but this method cannot be used to find the Moon's diameter. Fortunately help is at hand in both cases; we can measure angles and use these to calculate the values we require. In the case of the tree, the angle used is the angle between the ground (assumed to be horizontal) and a straight line drawn to the top of the tree; this angle is marked θ (the Greek letter theta) in Figure 6.1.

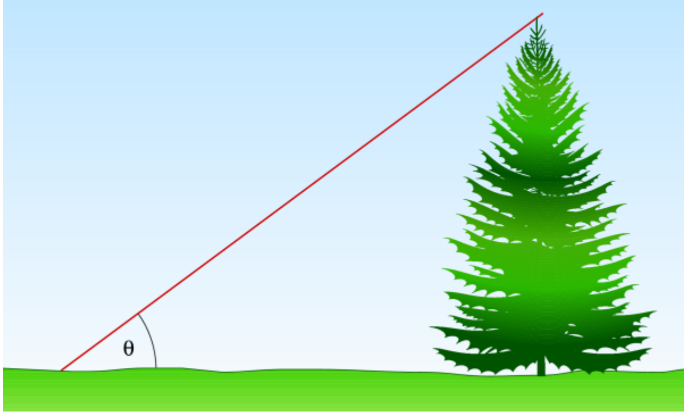


Figure 6.1: Chapter 6 will show how to use angles to find the height of a tree

Remember that a list of Greek letters and their pronunciation is given in Table 3.1.

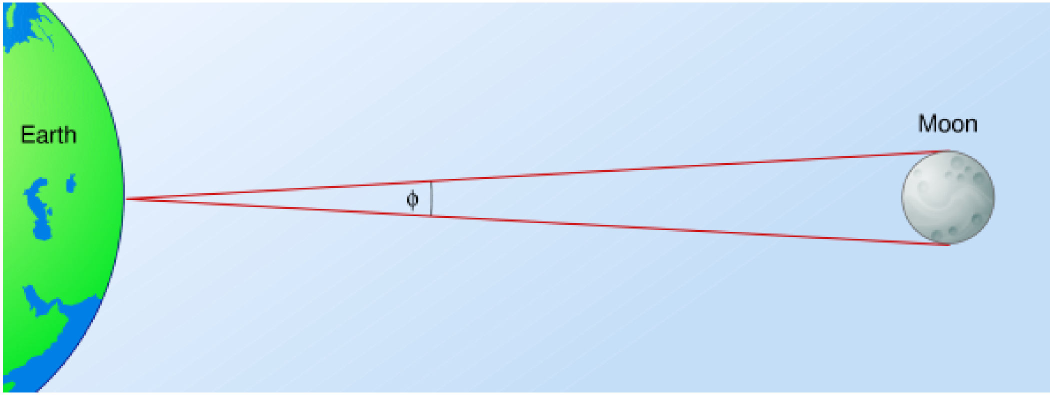


Figure 6.2: Chapter 6 will show how to use angles to find the diameter of the Moon

In the case of the Moon the angle is the one subtended (i.e. swept out) as a straight line drawn from an observer on the Earth moves from one side of the Moon to the other; this angle is labelled ϕ (the Greek letter phi) in Figure 6.2.

Section 6.1 describes two different systems used for measuring angles and, after a brief look at some of the properties of triangles, the rest of the chapter shows how angles can be used in scientific calculation to determine things such as the height of a tree and the diameter of the Moon.

6.1 Measuring angles: degrees and radians

You are probably familiar with the use of a protractor to measure angles shown on diagrams; this gives a result in degrees (represented by the symbol °, and sometimes known as 'degrees of arc' to make it clear that the degrees used to measure angles have nothing whatsoever to do with the degrees used when measuring temperature on the Celsius scale). Figure 6.3 shows that angle θ from Figure 6.1 is about 36.5°.

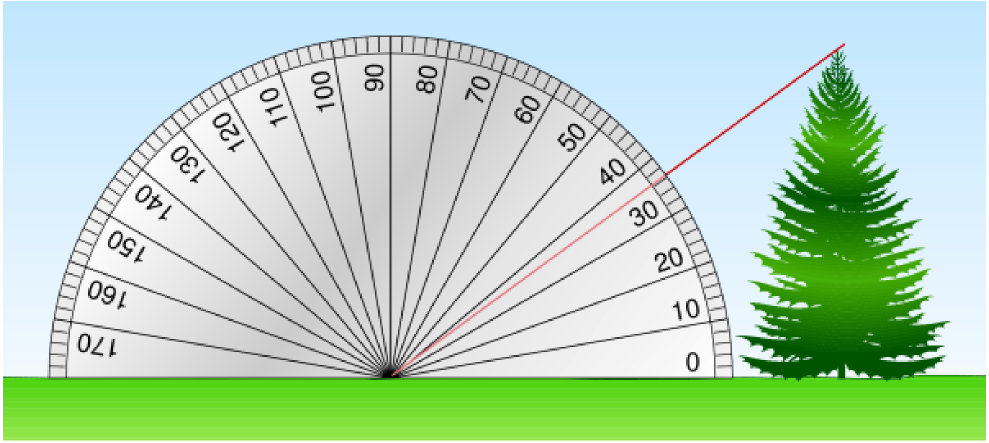


Figure 6.3: Measuring an angle with a protractor.

If you stand facing in a particular direction then turn through a complete revolution, you will have gone through 360°. The use of 360° to represent a complete turn is believed to date back to the ancient Babylonian civilization; 360 subdivisions were used because 360 is close to 365, the number of days in a year. Figure 6.4 illustrates various angles encountered in turning through a circle. Note in particular that a right angle (the angle between two directions that are perpendicular to each other) measures 90°.

Box 6.1 on the next page describes the use of angles to define lines of longitude and latitude on the Earth's surface, and hence to specify positions on the surface of the Earth.

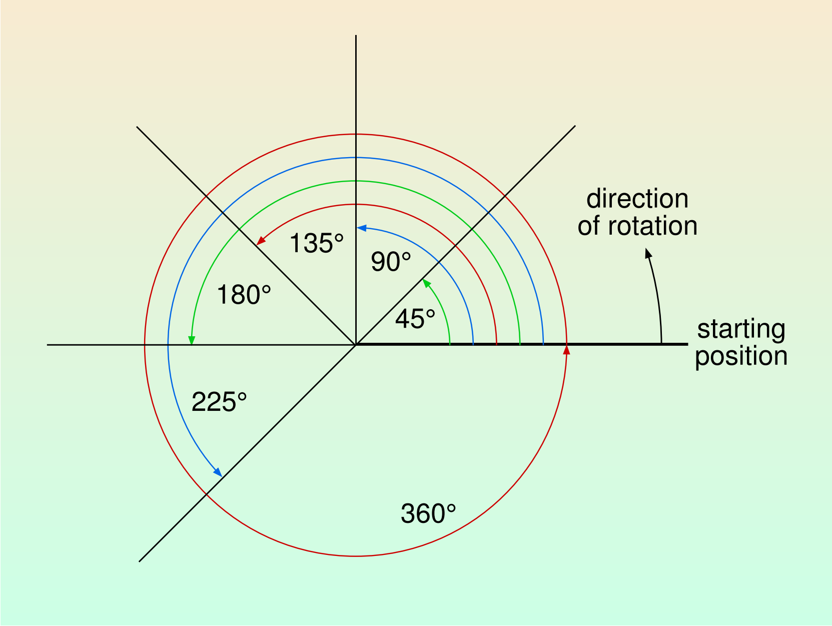


Figure 6.4: Angles encountered in turning through a circle.

Box 6.1 Lines of longitude and latitude

The surface of the Earth is conventionally marked with two sets of imaginary lines, as shown in Figure 6.5. The blue lines running from left to right in Figure 6.5 are lines of latitude, the Equator being one such line, and the red lines running from one pole to the other are lines of longitude.

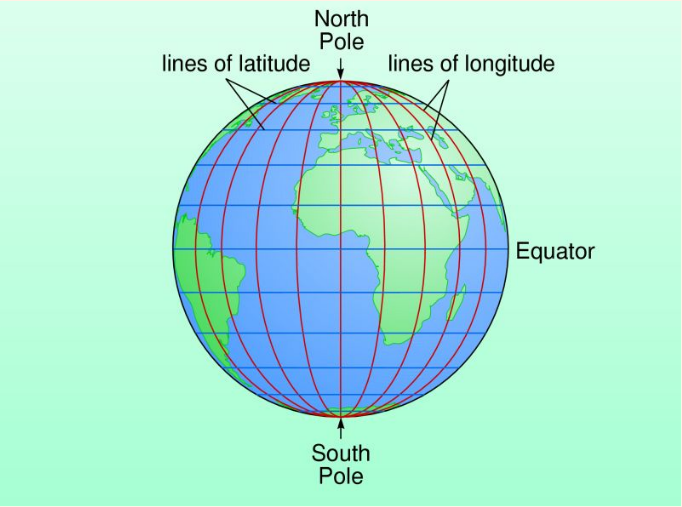


Figure 6.5: A model of the Earth viewed from above the Equator, showing lines of latitude and longitude.

In Figure 6.6, which is the view from above the North Pole, the circles are the lines of latitude and the lines radiating out from the pole are lines of longitude. It is easy to see, from Figure 6.6, how angles of longitude can be labelled using degrees. A line running through Greenwich in east London, and known as the Greenwich Meridian, is defined to be 0° longitude, and other lines are labelled by measuring the angles to the east or west of the Greenwich Meridian.

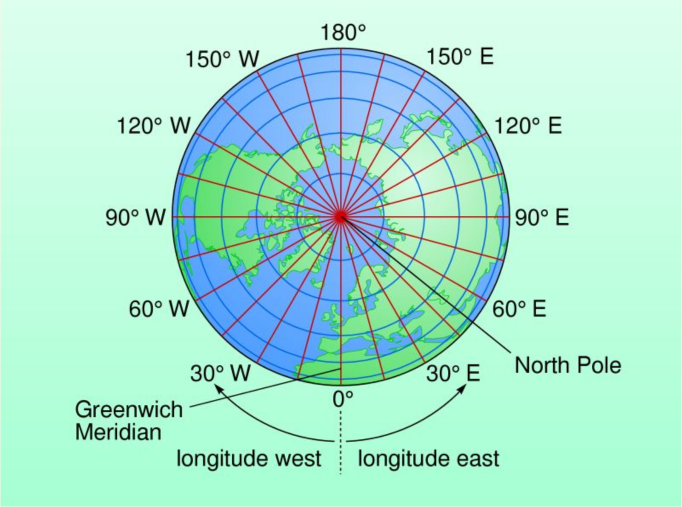


Figure 6.6: A model of the earth viewed from above the North Pole, showing lines of latitude and longitude.

Figure 6.7 shows how angles can be used to label lines of latitude too.

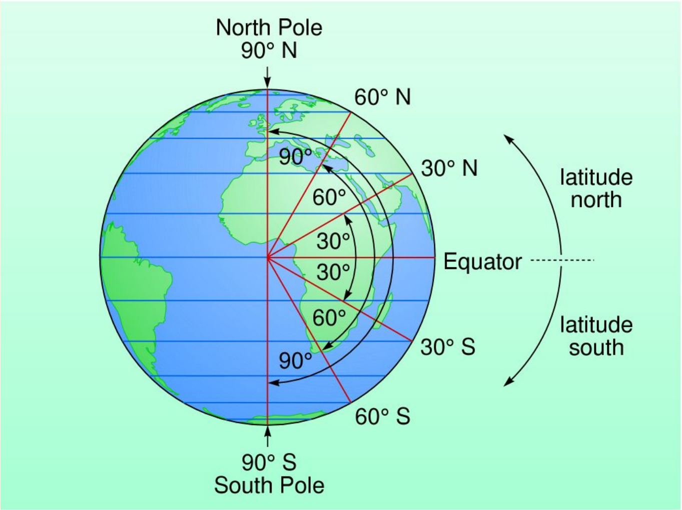


Figure 6.7: Using angles to label lines of latitude.

Figures 6.5, 6,6 and 6.7 show lines of longitude and latitude at 15° intervals only, but in reality the lines can be drawn as close together as required, and so can be used to specify a location very precisely. In order to specify a precise location, we need to subdivide degrees of longitude and latitude in some way. Historically this was done by dividing each degree into 60 minutes (or 'minutes of arc') in the same way as each hour is divided into 60 minutes (of time). The symbol ' ′ ' is used to represent minutes of arc. The longitude of Heathrow Airport (approximately 30 km west of Greenwich) is $0°27'$ W and both Greenwich and Heathrow have a latitude of about $51°28'$ N.

Minutes of arc are rarely used in modern science; small angles are usually expressed in decimal notation. Since $28'$ is 28 sixtieths of a degree and $\frac{28}{60} = 0.47$ to two significant figures, $51°28'$ can be written as $51.47°$. However, astronomers continue to use a further extension of the 'degrees and minutes' notation, simply because the angles they are measuring are frequently very small (since the objects they are measuring are such a long way from Earth). In order to measure such small angles, minutes of arc are further divided into 60 seconds of arc, or arcsecs (in the same way as minutes of time are subdivided into 60 seconds). So

$$1 \text{ arcsec} = \frac{1}{60} \text{ minute of arc} = \frac{1}{60} \times \frac{1}{60} \text{ degree} = \frac{1}{3600} \text{ degree}$$

As the Earth orbits the Sun, the next nearest star, Proxima Centauri, appears to move through an angle of 0.772 arcsecs across the sky; this is a mere $\frac{0.772}{3600}$ of a degree, i.e. 2.14×10^{-4} degrees.

Angles in science are frequently measured in radians rather than in degrees and subdivisions of degrees. Consider the circle shown in Figure 6.6. A part of the circumference, such as that between point X and point Y, is known as an arc, and in this case the arc subtends an angle θ. The length of the arc between X and Y is s and the radius of the circle is r. The radian is defined with reference to arc length and radius.

The size in radians of the angle, θ, subtended by an arc is defined to be arc length, s, divided by radius, r, thus

$$\theta \text{ (in radians)} = \frac{s}{r} \tag{6.1}$$

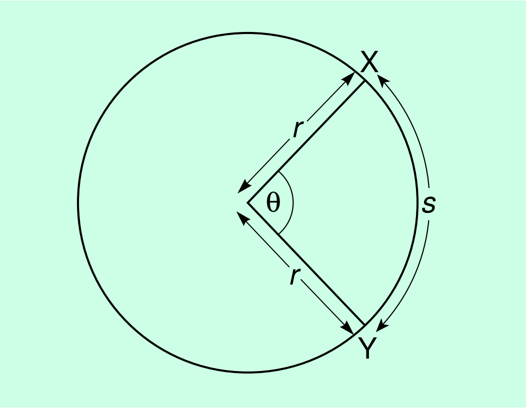


Figure 6.8: An arc of length s subtended by the angle θ in a circle of radius r.

Question

What is the size in radians of the angle subtended by an arc of length 3.0 cm in a circle of radius 2.0 cm?

Answer

From Equation 6.1 the angle is given by:

$$\begin{aligned}\theta \text{ in radians} &= \frac{s}{r}\\ &= \frac{3.0\text{ cm}}{2.0\text{ cm}}\\ &= 1.5\end{aligned}$$

So the size of the angle is 1.5 radians.

Note that since we have divided a length in centimetres by another length in centimetres, it could be argued that the answer should have no units. However, this course will adopt the common practice of writing the word ‘radians’ next to angles given in this measuring system, to distinguish them from angles measured in degrees or in any other system of angular measure.

An angle subtended by a longer arc in a circle of the same radius will be larger, as expected. In the above example, an arc of length 5.0 cm would subtend an angle of $\dfrac{5.0\text{ cm}}{2.0\text{ cm}}$, i.e. 2.5 radians.

Note, however, that it is the *ratio* of arc length to radius which is important in the definition of radian. This is illustrated in Figure 6.9, which shows two concentric circles (i.e. two circles with their centres at the same point).

The smaller circle has radius r, and an arc of length s (subtended by angle θ) is shown. In the larger circle, of radius r', the same angle θ is subtended by an arc of length s'. The superscript ‘ $'$ ’ is used to indicate that the lengths r' and s' (said as ‘r-prime and s-prime’, or ‘r-dash and s-dash’) both relate to the same circle. The lengths s' and r' are bigger than the values of s and r, as you would expect, but the ratios $\dfrac{s}{r}$ and $\dfrac{s'}{r'}$ are *equal*, and the angle subtended in radians is

$$\theta = \frac{s}{r} = \frac{s'}{r'}$$

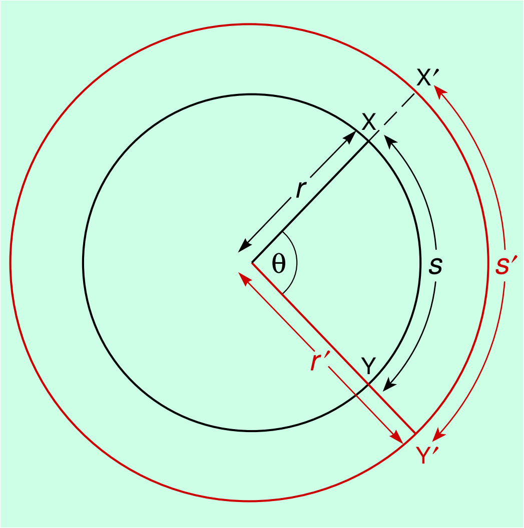


Figure 6.9: Two concentric circles.

Let's now consider two special cases. In the first, the arc length is exactly equal to the radius, as shown in Figure 6.10a, i.e. $s = r$. This means that

$$\theta \text{ (in radians)} = \frac{s}{r} = \frac{r}{r} = 1$$

i.e. the angle subtended is one radian.

In the second special case, illustrated in Figure 6.10b, the arc length is a complete circumference. For all circles, the circumference, C, is linked to the radius, r, by the formula $C = 2\pi r$ (this formula, given in Box 3.4, follows directly from the definition of π, given in Section 1.1.1 as circumference divided by diameter). So when the arc length, s, is equal to the whole circumference, C, $s = 2\pi r$ so

$$\theta \text{ (in radians)} = \frac{2\pi r}{r} = 2\pi$$

Thus the angle subtended by a complete revolution is 2π radians, i.e. 2π radians = 360°.

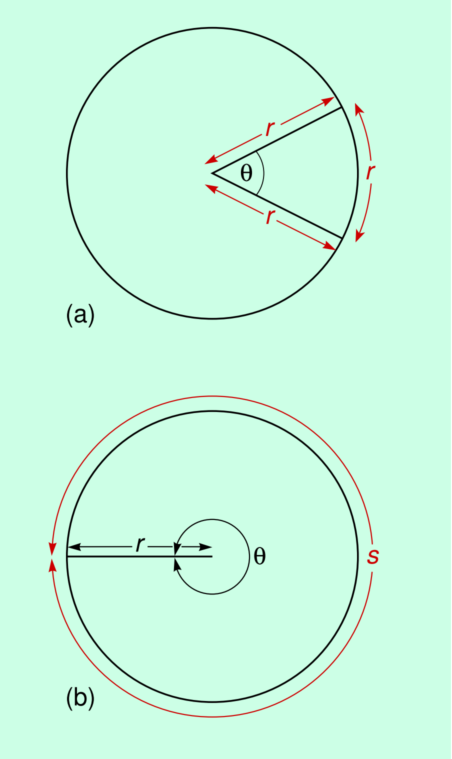


Figure 6.10: The angle subtended when (a) arc length is equal to radius, and (b) arc length is equal to circumference.

This gives us an easy way of converting between degrees and radians.

Since 2π radians = 360°,

$$\begin{aligned} 1 \text{ radian} &= \frac{360^\circ}{2\pi} \\ &= \frac{180^\circ}{\pi} \\ &\approx 57.3^\circ \end{aligned}$$

where the symbol '$\approx$' means 'is approximately equal to', as in Chapter 3.

Similarly, since $360^\circ = 2\pi$ radians,

$$\begin{aligned} 1^\circ &= \frac{2\pi}{360} \\ &= \frac{\pi}{180} \\ &\approx 0.0175 \text{ radians} \end{aligned}$$

Note that the numerical conversion factors between radians and degrees are only approximate (they have been given to three significant figures), so when converting from radians to degrees or vice versa it is best to go back to first principles in each case, remembering that a complete revolution can be represented by either 2π radians or 360°.

It is also worth remembering that angles in radians are frequently expressed as multiples or fractions of π so, for example,

$$\begin{aligned} 45^\circ &= 45 \times \frac{2\pi}{360} \text{ radians} \\ &= \frac{\pi}{4} \text{ radians} \end{aligned}$$

An angle of exactly 45° is equal to *exactly* $\frac{\pi}{4}$ radians.

Worked example 6.1

Express $\frac{\pi}{6}$ radians in degrees.

Answer

2π radians = 360° so π radians = 180°.

$$\frac{\pi}{6} \text{ radians} = \frac{180^\circ}{6} = 30^\circ.$$

Worked example 6.2

The angle subtended as a straight line drawn from an observer on the Earth moves from one side of the Moon to the other is 0.519°. (This is angle ϕ in Figure 6.2, but remember that the figure is not drawn to scale). Express this angle in radians.

Answer

$$360^\circ = 2\pi \text{ radians} \quad \text{so} \quad 1^\circ = \frac{2\pi}{360} \text{ radians}$$

$$\begin{aligned} 0.519^\circ &= 0.519 \times \frac{2\pi}{360} \text{ radians} \\ &= 9.06 \times 10^{-3} \text{ radians to three significant figures.} \end{aligned}$$

Question 6.1

Convert the following from radians to degrees:

(a) 0.123 radians — Answer

(b) $\frac{2\pi}{3}$ radians — Answer

(c) $\frac{3\pi}{2}$ radians — Answer

Question 6.2

Convert the following from degrees to radians:

(a) $36.5°$ (angle θ in Figure 6.1) Answer

(b) $90°$ Answer

(c) $210°$ Answer

6.2 A quick look at triangles

Note the labelling system used for angles in Figure 6.11. Angle α could also be identified as angle BAC, $\angle$BAC or $\hat{\mathrm{A}}$, but in this course angles will be labelled on the *inside* in the way angles α, β and γ have been labelled in Figure 6.11. If you measure the size of the angles inside the triangle shown in Figure 6.11 with a protractor, you will find that $\alpha = 80°$, $\beta = 60°$ and $\gamma = 40°$. Thus

$$\alpha + \beta + \gamma = 80° + 60° + 40° = 180°$$

This result is true for all triangles, i.e.

For all triangles, the internal angles add up to 180°.

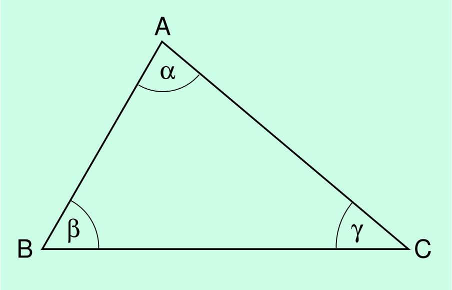


Figure 6.11: The angles inside a triangle.

If you wish, you can check that this result holds for all of the triangles shown in Figure 6.12, irrespective of the shape of the triangle. Figure 6.12e and Figure 6.12f illustrate two triangles of a particular type; each has one internal angle equal to 90°, i.e. a right angle, so they are known as right-angled triangles. Note that the right angles have been labelled in the conventional way, with a square corner. In a right-angled triangle, since the internal angles must total 180° and one of the three angles is 90°, it follows that the other two angles must add up to a total of 90° too. This result means that if you know that a triangle is right-angled, and you know one of the other angles, you can find the remaining angle without needing to measure it. In Figure 6.12e, $\alpha = 30°$, so $\beta = 90° - 30° = 60°$.

Pythagoras' Theorem, whose proof is accredited to the Greek philosopher Pythagoras or one of his followers about 2500 years ago, but which was probably known even earlier, gives us a way of calculating the length of a third side of a right-angled triangle from a knowledge of the lengths of the other two sides.

The side opposite the right angle in a right-angled triangle is known as the hypotenuse, and Pythagoras' Theorem is commonly stated as

> The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.

In the triangle shown in Figure 6.13, the hypotenuse has a length h and the other two sides have lengths a and b. Thus

$$h^2 = a^2 + b^2 \tag{6.2}$$

We are only interested in the positive square root, so

$$h = \sqrt{a^2 + b^2}$$

If $a = 3$ cm and $b = 4$ cm in a right-angledtriangle, then

$$\begin{aligned} h &= \sqrt{a^2 + b^2} = \sqrt{(3\text{ cm})^2 + (4\text{ cm})^2} = \sqrt{9\text{ cm}^2 + 16\text{ cm}^2} \\ &= \sqrt{25\text{ cm}^2} = 5\text{ cm} \end{aligned}$$

If $h = 9.1$ m and $a = 5.1$ m in a different right-angled triangle, then $h^2 = a^2 + b^2$ can be rearranged to give $b^2 = h^2 - a^2$ so

$$\begin{aligned} b &= \sqrt{h^2 - a^2} = \sqrt{(9.1\text{ m})^2 - (5.1\text{ m})^2} = \sqrt{82.81\text{ m}^2 - 26.01\text{ m}^2} \\ &= \sqrt{56.80\text{ m}^2} = 7.5\text{ m to 2 significant figures.} \end{aligned}$$

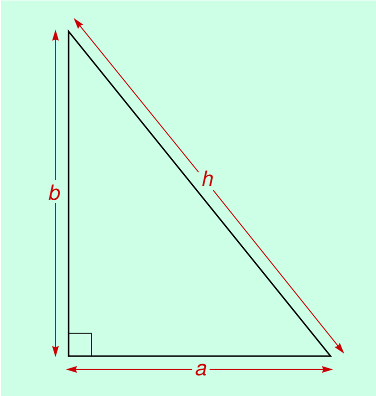


Figure 6.13: A right-angled triangle.

Question 6.3

The base of ladder of length 4.50 m is placed on level ground at a distance of 1.15 m from a vertical wall, and the top of the ladder leans against the wall. The angle between the ground and the ladder is found to be 75.2°. Calculate

(a) the height that the ladder reaches up the wall; Answer

(b) the angle between the wall and the top of the ladder. Answer

Hint: you may find it helpful to start by drawing a diagram of the situation.

Pythagoras' Theorem provides us with a way of finding unknown lengths from known lengths; the fact that the internal angles in any triangle add up to 180° provides us with a way of finding unknown angles from known angles. Trigonometry, discussed in Section 6.3, takes us one stage further by providing a way of finding unknown lengths from known angles and unknown angles from known lengths.

6.3 Calculating with angles: trigonometry

Trigonometry is the branch of mathematics that deals with the relationships between the sides and angles of triangles. The Greek astronomer Hipparchus is credited with its invention, but the principles involved were almost certainly in use even earlier by the ancient Egyptians surveying the land surrounding the Nile. Despite its ancient origins, trigonometry plays an important part in modern science.

Look at the three right-angled triangles shown in Figure 6.14. These triangles are similar, i.e. they have the same shape (although their sizes are different); note in particular that the angle θ is exactly the same in each of the three triangles.

The superscript symbols ‘ $'$ ’ and ‘ $''$ ’ (‘prime and double-prime’ or ‘dash and double-dash’) indicate lengths relating to the second and third triangles respectively.

As you look at Figure 6.14 from left to right, you will see that the triangles have sides of increasing length; however the ratio of any one side to each of the other sides remains constant, thus

$$\frac{b}{a} = \frac{b'}{a'} = \frac{b''}{a''} \tag{6.3}$$

$$\frac{b}{h} = \frac{b'}{h'} = \frac{b''}{h''} \tag{6.4}$$

$$\frac{a}{h} = \frac{a'}{h'} = \frac{a''}{h''} \tag{6.5}$$

If the angle θ and hence the shape of the triangle had been different, the ratios would have had different values. Thus each angle θ gives rise to unique values for $\dfrac{b}{a}$, $\dfrac{b}{h}$ and $\dfrac{a}{h}$, and conversely each value for $\dfrac{b}{a}$, $\dfrac{b}{h}$ or $\dfrac{a}{h}$ in a triangle leads to a particular value for θ. This result is so important that the ratios are given the special names tangent, sine and cosine, usually abbreviated to tan, sin and cos. Tan, sin and cos are known collectively as trigonometric (or trig.) ratios.

The tangent of angle θ is defined by

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \tag{6.6}$$

This is the ratio we have been describing as $\frac{b}{a}$, where b is the side opposite angle θ and a is the side (other than the hypotenuse) that is adjacent (next to) angle θ.

The sine of angle θ is defined by

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \tag{6.7}$$

This is the ratio we have been describing as $\frac{b}{h}$.

The cosine of angle θ is defined by

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{6.8}$$

This is the ratio we have been describing as $\frac{a}{h}$.

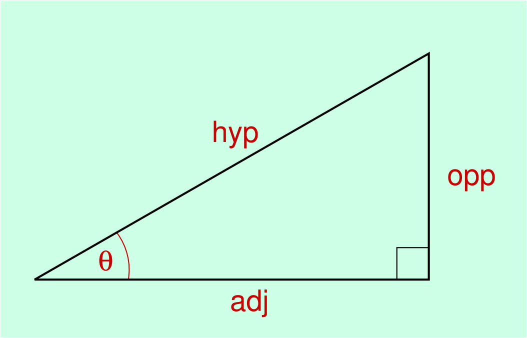


Figure 6.15: $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$; $\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$; $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$

The sides opposite and adjacent to a particular angle in a right-angled triangle are usually abbreviated to 'opp' and 'adj' and the hypotenuse is abbreviated to 'hyp', as shown in Figure 6.15.

Note that the trigonometric ratios are defined with respect to a particular angle in a right-angled triangle. If we had considered the other non right-angled angle in the triangle in Figure 6.15, the 'opposite' and 'adjacent' sides would have been different, and so the sine, cosine and tangent would have been different too.

The trigonometric ratios were tabulated many years ago and generations of scientists have used tables and slide rules similar to those shown in Figure 6.16 to calculate lengths from angles and angles from lengths. Nowadays, trigonometric ratios are available at the press of a calculator button.

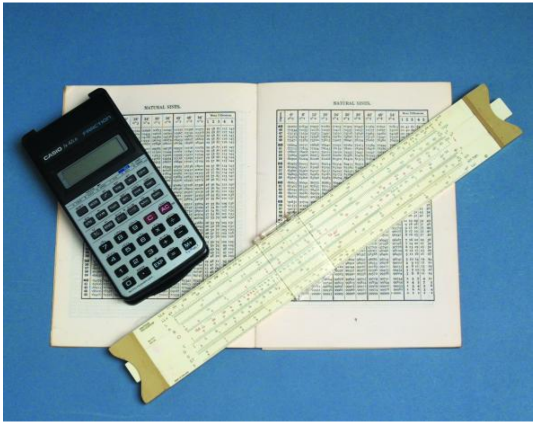


Figure 6.16: Tables, slide rules and calculators can all be used to find trigonometric ratios.

6.3.1 Using a calculator for trigonometry

Make sure that you can use your calculator to find trigonometric ratios. The sine, cosine and tangent functions are likely to be clearly marked as 'sin', 'cos' and 'tan'. Remember, from Section 6.1, that angles can be measured in either degrees or radians. Your calculator should be able to cope with either of these (and possibly a third angular measure called 'grad' too) but you need to ensure that the calculator is in the correct 'mode'. Angle θ in Figure 6.15 is $30°$, alternatively written as $\dfrac{\pi}{6}$ radians, so the sine of angle θ could be expressed as either $\sin 30°$ or $\sin\dfrac{\pi}{6}$ (where $\dfrac{\pi}{6}$ is in radians, though the word 'radians' is usually omitted when finding trigonometric ratios). Note that $\sin\dfrac{\pi}{6}$ (which is the sine of the angle $\dfrac{\pi}{6}$ and could be written as $\sin\left(\dfrac{\pi}{6}\right)$ for clarity) is not the same as $\dfrac{\sin\pi}{6}$ (which is $\dfrac{1}{6}$th of the sine of the angle π and could be written as $\dfrac{(\sin\pi)}{6}$ for

clarity).

Check that you can use your calculator to give:

$$\sin 30^\circ = 0.5;\ \cos 30^\circ = 0.8660;\ \tan 30^\circ = 0.5774$$

and also to give:

$$\sin\frac{\pi}{6} = 0.5;\ \cos\frac{\pi}{6} = 0.8660;\ \tan\frac{\pi}{6} = 0.5774$$

where the answers are either exact or given to four significant figures.

Note that when using trigonometric ratios you should always work to at least four significant figures (although you should round your answer to an appropriate number of significant figures at the end of a calculation).

Question 6.4

Use your calculator to find:

(a) $\sin 49^\circ$ Answer

(b) $\cos\frac{\pi}{8}$ (where $\frac{\pi}{8}$ is in radians) Answer

(c) $\tan\frac{\pi}{4}$ (where $\frac{\pi}{4}$ is in radians). Answer

You will also need to be able to use your calculator to find the angle which has a particular sine, cosine or tangent. For example, if you know that $\tan\theta = 0.75$, then what is θ in degrees? What you are looking for is known as the 'inverse tangent' or arctangent and you need to use a button on your calculator labelled as '$\tan^{-1}$' or 'arctan'. Check that you can use your calculator to give the correct answer, which is that $\tan^{-1}(0.75) = 37^\circ = 0.64$ radians to two significant figures. Your calculator should also be able to calculate 'inverse sine' (using a button labelled as '$\sin^{-1}$' or arcsin and 'inverse cosine' ('$\cos^{-1}$' or arccos). Note that '$\tan^{-1}$', '$\sin^{-1}$' and '$\cos^{-1}$' are properly referred to as the inverse functions of tan, sin and cos (as they work in the opposite direction) but care needs to be taken to avoid confusion with reciprocals:

$$\tan^{-1} \neq \frac{1}{\tan}$$

$$\sin^{-1} \neq \frac{1}{\sin}$$

$$\cos^{-1} \neq \frac{1}{\cos}$$

remembering that $\neq$ means 'is not equal to'.

Question 6.5

Use your calculator to find:

(a) Use your calculator to find the angle α (in degrees) for which $\cos\alpha = 0.5253$. Answer

(b) Use your calculator to find the angle β (in radians) for which $\tan\beta = 1.5574$. Answer

Note that although we have only defined trigonometric ratios for angles in a right-angled triangle, and most of the angles for which trigonometric ratios are used in this course are acute (i.e. less than 90°), values of sin, cos and tan can be found for larger angles too. Use your calculator to check that $\sin\pi = 0$, $\cos\pi = -1$ and $\tan\pi = 0$ (where π is an angle in radians, equal to 180°). Box 6.2 considers the sines and cosines of angles greater than 90° in slightly more detail, and it also introduces you to *negative* angles and their trigonometric ratios.

Box 6.2 Using trigonometric ratios to describe waves

It is possible to assign values for $\sin\theta$ and $\cos\theta$ for *all* angles, however large they are. Table 6.1 gives values for $\sin\theta$ and $\cos\theta$ for selected values of θ up to the arbitrarily chosen value of 3π (540°). The angles are like those encountered in Figure 6.4 in turning through a complete circle, except that there is no need to stop at 360°, and the angles are now measured in radians.

Two results have been omitted from Table 6.1.

Question

Use your calculator to find the sine of $\dfrac{3\pi}{2}$ radians (270°) and the cosine of $\dfrac{13\pi}{6}$ radians (390°) and add these values to Table 6.1.

Answer

$$\sin\frac{3\pi}{2} = -1$$

$$\cos\frac{13\pi}{6} = 0.8660 \text{ to four significant figures.}$$

θ in radians	$\sin\theta$	$\cos\theta$
0	0	1
$\pi/6$	0.5	0.8660
$\pi/3$	0.8660	0.5
$\pi/2$	1	0
$2\pi/3$	0.8660	−0.5
$5\pi/6$	0.5	−0.8660
π	0	−1
$7\pi/6$	−0.5	−0.8660
$4\pi/3$	−0.8660	−0.5
$3\pi/2$		0
$5\pi/3$	−0.8660	0.5
$11\pi/6$	−0.5	0.8660
2π	0	1
$13\pi/6$	0.5	
$7\pi/3$	0.8660	0.5
$5\pi/2$	1	0
$8\pi/3$	0.8660	−0.5
$17\pi/6$	0.5	−0.8660
3π	0	−1

Table 6.1: Values of $\sin\theta$ and $\cos\theta$ for θ from 0 to 3π

If instead of turning in an anticlockwise direction in the way used to define angles up to 360°and beyond (Figure 6.4), we had turned in a clockwise direction, the angles would have been measured in the opposite direction. Angles such as these are defined to be negative e.g. $-\dfrac{\pi}{6}$, $-\pi$, $-\dfrac{3\pi}{2}$. Values for $\sin\theta$ and $\cos\theta$ can also be assigned for negative values of θ, as shown in Table 6.2.

Inspection of Table 6.1 and Table 6.2 shows that $\sin\theta$ and $\cos\theta$ each vary between -1 and $+1$ across the whole range of values for θ. The form of the variation is made clearer by the graphs shown in Figure 6.17.

The graphs may remind you of the sort of wave pattern observed when you take an instantaneous sideways look at waves on a pond. In fact, sine and cosine functions, of the form $y = a\sin\theta$ and $y = a\cos\theta$ (where a is a constant) are extensively used in describing the motion of waves of all types. The detail is beyond the scope of this course, but it is another application of maths in science!

θ in radians	$\sin\theta$	$\cos\theta$
0	0	1
$-\pi/6$	-0.5	0.8660
$-\pi/3$	-0.8660	0.5
$-\pi/2$	-1	0
$-2\pi/3$	-0.8660	-0.5
$-5\pi/6$	-0.5	-0.8660
$-\pi$	0	-1
$-7\pi/6$	0.5	-0.8660
$-4\pi/3$	0.8660	-0.5
$-3\pi/2$	1	0
$-5\pi/3$	0.8660	0.5
$-11\pi/6$	0.5	0.8660
-2π	0	1
$-13\pi/6$	-0.5	0.8660
$-7\pi/3$	-0.8660	0.5
$-5\pi/2$	-1	0
$-8\pi/3$	-0.8660	-0.5
$-17\pi/6$	-0.5	-0.8660
-3π	0	-1

Table 6.2: Values of $\sin\theta$ and $\cos\theta$ for θ from 0 to -3π

6.3.2 Using trigonometry in science

We are now in a position to be able to find the height of the tree mentioned at the beginning of the chapter. This is shown as H in Figure 6.18. We know that $\theta = 36.5°$ and suppose we have measured D, the distance to the tree from the point at which the angle θ was measured, to be 28.6 m. How tall is the tree?

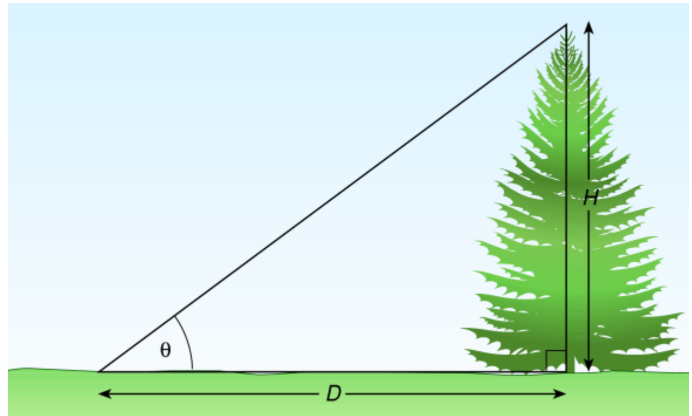


Figure 6.18: Using trigonometry to find the height of a tree. $\theta = 36.5°$ and $D = 28.6$ m.

We can say that

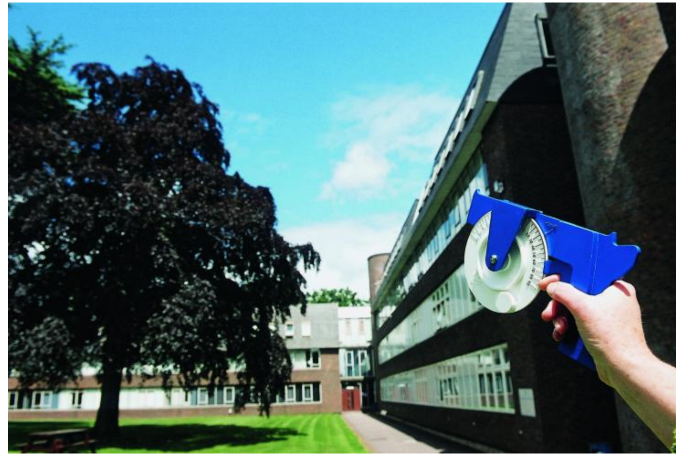
$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{H}{D} \text{ in this case.}$$

We need to rearrange this equation to make H the subject; we can do this in exactly the same way as we did in Chapter 4, by reversing the equation and then multiplying both sides of the equation by D. This gives

$$H = D\tan\theta$$

So $H = 28.6\text{ m} \times \tan 36.5° = 21.2$ m to 3 significant figures.

It was clearly stated at the beginning of the chapter that θ was the angle between the *ground* and a straight line drawn to the top of the tree, but in reality you're more likely to have taken readings at eye level, perhaps using an instrument such as a 'gun clinometer', whose use is illustrated in Figure 6.19. The gun clinometer measures the angle shown as α in Figure 6.19b, and Worked example 6.3 shows how this can be used to find the height of a tree.



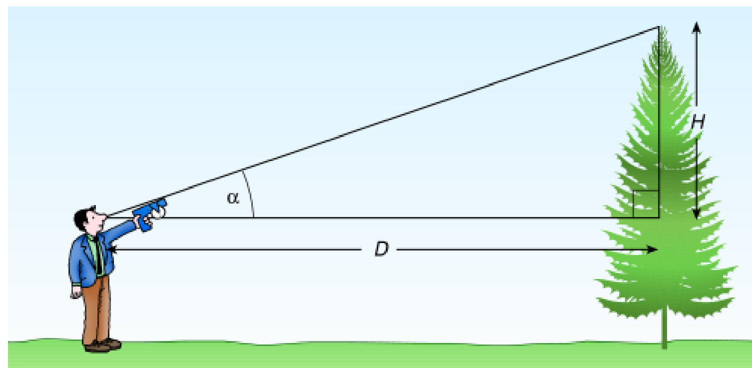


Figure 6.19: (a) Using a gun clinometer to find the height of a tree; (b) the gun clinometer gives angle α.

Question

When used by a man of height 1.8 m and in the way illustrated in Figure 6.19, a gun clinometer records an angle α of 39° at a distance, D, of 18 m from a tree. What is the height of the tree?

Answer

$\tan\alpha = \dfrac{H}{D}$ where $\alpha = 39°$ and $D = 18$ m, so

$$\begin{aligned} H &= D\tan\alpha \\ &= 18\text{ m} \times \tan 39° \\ &= 14.6\text{ m} \end{aligned}$$

On this occasion, however, the reading was taken at eye level, so H is *not* the height of the tree. Assuming that it is 1.7 m from the ground to the man's eyes and that the ground is horizontal, the height of the tree is 1.7 m more than H, i.e. the height of the tree is 16 m to two significant figures.

Question 6.7 asks you to use trigonometry in solving another simulated 'real world' problem, but Question 6.6 is given first to enable you to practise the underlying trigonometric and algebraic skills.

Question 6.6

(a) Find length h in Figure 6.20a. Answer

(b) Find length a in Figure 6.20b. Answer

(c) Find angle θ in Figure 6.20c, giving your answer in degrees. Answer

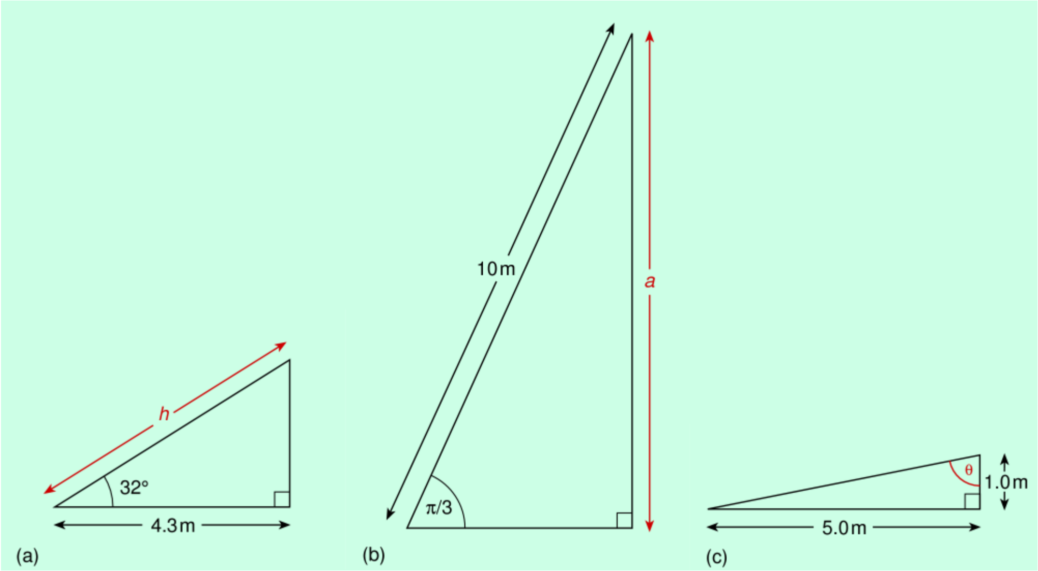


Figure 6.20: Right-angled triangles for use in Question 6.6 (*not drawn to scale*).

Question 6.7 Answer

A theodolite of height 1.5 m is positioned with its base at sea-level somewhere in the Cambridgeshire Fens, and indicates that the top of Ely Cathedral's West Tower is at an inclination of 2.27° (see Figure 6.21). The base of Ely Cathedral is 15 m above sea-level and the West Tower is 66 m tall. Approximately how far is the theodolite from Ely Cathedral?

Hint: start by finding H, the vertical distance between the top of the theodolite and the top of the West Tower.

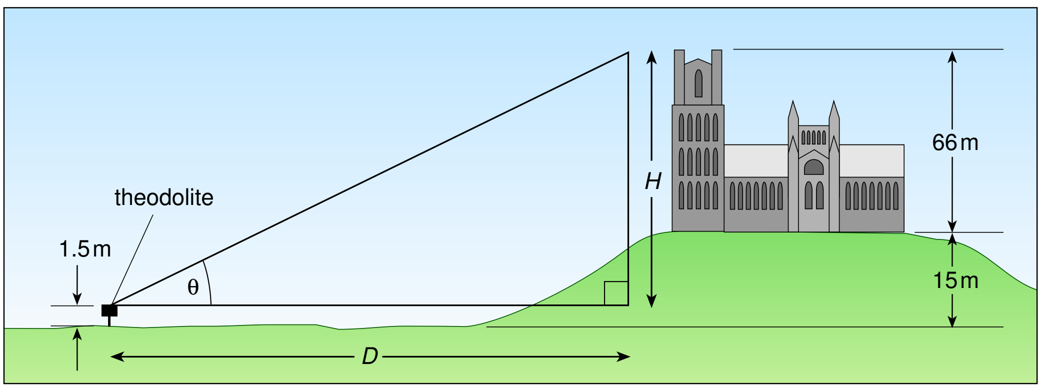


Figure 6.21: Using trigonometry to find distance *(not to scale)*. $\theta = 2.27°$.

In addition to providing a way of finding unknown lengths and angles, trigonometric ratios appear from time to time in scientific equations. You are not expected to remember these equations or to understand the background science; brief explanations are provided in Boxes 6.3–6.6 for interest only.

Box 6.3 Angle of dip and true thickness of strata

Folding and tilting of layers of rocks, caused by pressures within the Earth, have resulted in many layers lying at an angle to the Earth's surface. This angle is called the angle of dip and is illustrated in Figure 6.22. The angle of dip can usually be measured, as can the apparent width of a stratum (layer) at the Earth's surface — its outcrop, but it is the *true* thickness of the stratum which is of real interest to geologists.

The vertical thickness of the stratum (in Figure 6.23) may also be of interest, especially when exploring for underground resources (such as oil) by drilling.

Question

Express $\sin\theta$ in Figure 6.22 in terms of T and W. Hence find an equation for the true thickness, T, of a stratum in terms of the width, W, of the outcrop at the Earth's surface, and the angle of dip, θ.

Answer

$$\sin\theta = \frac{T}{W}$$

so

$$T = W\sin\theta \tag{6.9}$$

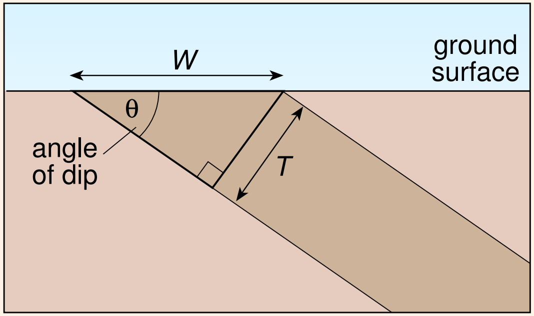


Figure 6.22: The relationship between the angle of dip, θ, width of outcrop, W, and true thickness, T, for a tilting stratum of rock (shown in darker brown).

Question

Express $\tan\theta$ in Figure 6.23 in terms of V and W. Hence find an equation for the vertical thickness, V, of a stratum in terms of the width, W, of the outcrop at the Earth's surface, and the angle of dip, θ.

Answer

$$\tan\theta = \frac{V}{W}$$

so

$$V = W\tan\theta \tag{6.10}$$

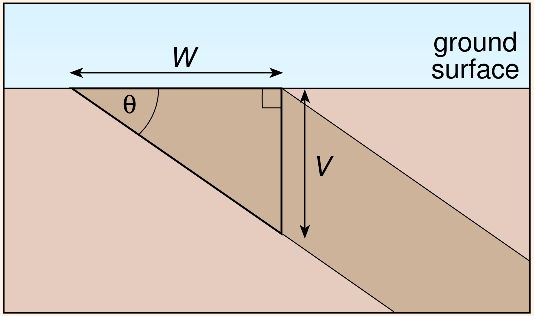


Figure 6.23: The relationship between θ, W, and the vertical thickness, V, of the stratum.

Worked example 6.4

Suppose a stratum of rock, lying at an angle of dip of 28°, has an outcrop of width of 71 m at the Earth's surface. What is its true thickness?

Answer

From Equation 6.9, $T = W \sin\theta$ where $W = 71$ m and $\theta = 28°$, so

$$T = 71\ \text{m} \times \sin 28^\circ = 33\ \text{m to two significant figures.}$$

The true thickness of the layer is 33 metres.

Question 6.8 Answer

What is the vertical thickness of a stratum of rock which has outcrop of width 65 m at the Earth's surface, and an angle of dip of 36°?

Hint: you should use an appropriate equation from Box 6.3.

Box 6.4 Using trigonometry to determine the radius of ions

The crystal structure of lithium iodide consists of lithium and iodide ions (ions are atoms with electric charge due to the loss or gain of electrons), as shown in Figure 6.24. Both types of ions can be represented by spheres and, in one model, the spheres can be considered just to touch each other. This enables us to use trigonometry to find the radius of the ions.

If the distance between the centre of a lithium ion and the centre of an iodide ion is known (this is the so-called inter-nuclear distance, and is labelled as h on Figure 6.24) then

$$\cos 45^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{r}{h}$$

where r is the radius of a lithium ion.

Multiplying both sides by h gives

$$r = h\cos 45^\circ \qquad (6.11)$$

Equation 6.11 can be used to find the radius of a lithium ion.

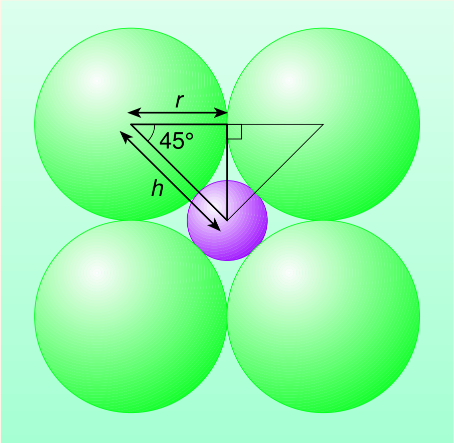


Figure 6.24: Using trigonometry to find the radius of lithium ions (shown in green). The purple sphere represents an iodide ion.

Question 6.9 Answer

The internuclear distance, *h*, between the ions shown in Figure 6.24 is measured to be 302 pm (where 1 pm = 10^{-12} m, as defined in Box 2.2).

Use Equation 6.11 to find the radius of a lithium ion.

Box 6.5 Snell's law for seismic waves and light

You may have realized, from the equation for S wave speed, $v_s = \sqrt{\frac{\mu}{\rho}}$ (much used in Chapters 3 and 4), that waves travel at different speeds in different substances. When a wave moves from one substance into another in which it travels at a different speed, the change in speed will cause the wave to change *direction*. This behaviour is known as refraction and it is illustrated in Figure 6.25.

Snell's law of refraction states that

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2} \tag{6.12}$$

where v_1 is the speed of the wave in the first substance, v_2 is the speed of the wave in the second substance, and i and r are the angles of incidence and refraction respectively, as illustrated in Figure 6.25.

Refraction occurs for all types of waves, for example, seismic waves passing from one rock type to another in the Earth, or a beam of light passing from air to glass, and Snell's Law is true whatever type of wave motion is being considered. The law is named after the Dutch scientist Willebrord Snel (1596–1650) but the law was stated very much earlier, by the mathematician Abu Said al-Ala Ibn Sahl in his book *On the Burning Instruments*, written in about 984.

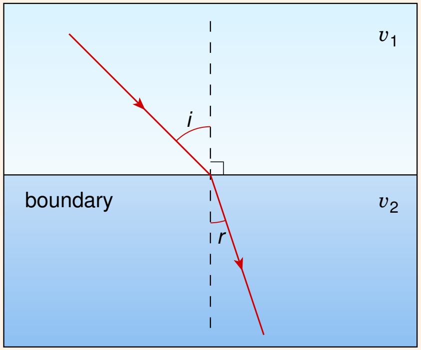


Figure 6.25: A wave undergoing refraction on passing through a boundary between two media in which the speeds of propagation, v_1 and v_2, are different (in this case $v_1 > v_2$); i is called the angle of incidence and r is called the angle of refraction.

Worked example 6.5

Calculate the angle of refraction of a seismic wave which has an angle of incidence of 35° on crossing a boundary from rock 1 (with $v_1 = 6.3 \times 10^3\ \mathrm{m\,s^{-1}}$) into rock 2 (with $v_2 = 8.2 \times 10^3\ \mathrm{m\,s^{-1}}$).

Answer

We know $v_1 = 6.3 \times 10^3\ \mathrm{m\,s^{-1}}$, $v_2 = 8.2 \times 10^3\ \mathrm{m\,s^{-1}}$ and $i = 35°$, and we want r. Snell's law states that

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2}$$

Multiplying both sides by $\sin r$ gives

$$\sin i = \frac{v_1}{v_2} \times \sin r$$

Reversing the equation and multiplying both sides by v_2 gives

$$v_1 \sin r = v_2 \sin i$$

Dividing both sides by v_1 gives

$$\sin r = \frac{v_2 \sin i}{v_1}$$

Substituting gives

$$\sin r = \frac{8.2 \times 10^3\ \mathrm{m\,s^{-1}} \times \sin 35^\circ}{6.3 \times 10^3\ \mathrm{m\,s^{-1}}}$$
$$= 0.7466$$

So

$$r = \sin^{-1}(0.7446)$$
$$= 48^\circ \text{ to 2 significant figures.}$$

Note that in this case the angle of refraction is greater than the angle of incidence. This is because v_2 is greater than v_1.

Question 6.10 Answer

A beam of light strikes an air–glass interface with an angle of incidence of 45.0° and the angle of refraction (in the glass) is found to be 26.3°. The speed of light in air is $3.00 \times 10^8\ \mathrm{m\,s^{-1}}$. Use Snell's law to find the speed of light in glass.

Box 6.6 Using a diffraction grating

A diffraction grating is simply a series of extremely narrow, evenly spaced slits through which light can pass. When a light beam of a single colour (i.e. a single wavelength) hits the diffraction grating at an angle of 90°, as shown in Figure 6.26, the grating acts in such a way as to split up the incoming beam, forming what is called a diffraction pattern. Some light passes straight through the grating; this is called the zero-order beam. Other beams are produced at angles θ_1, θ_2, etc. from the straight-through position and are known as the 1st, 2nd, etc. order diffracted beams.

The angle θ_n of the nth order beam is given by the equation

$$\sin\theta_n = \frac{n\lambda}{d} \tag{6.13}$$

where λ is the wavelength of the light and d is the grating spacing (i.e. the distance between two adjacent slits in the grating).

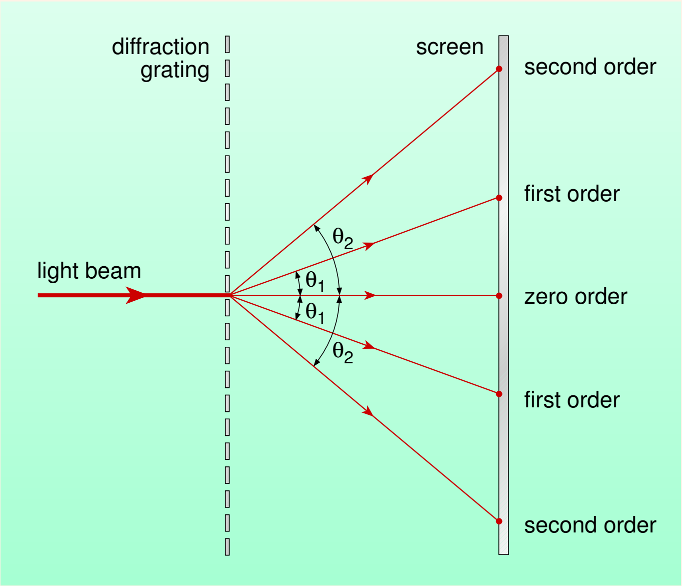


Figure 6.26: The pattern formed by a diffraction grating.

Worked example 6.6

A beam of light of wavelength 5.89×10^{-7} m passes through a diffraction grating and the second-order diffracted beam is at $\theta_2 = 45.9°$. Find the grating spacing d.

Answer

In this case $\lambda = 5.89 \times 10^{-7}$ m, $n = 2$, $\theta_n = 45.9°$.

Multiplying both sides of Equation 6.13 by d gives

$$d \sin\theta_n = n\lambda$$

Dividing both sides by $\sin\theta_n$ gives

$$\begin{aligned} d &= \frac{n\lambda}{\sin\theta_n} \\ &= \frac{2 \times 5.89 \times 10^{-7}\ \text{m}}{\sin 45.9°} \\ &= 1.64 \times 10^{-6}\ \text{m} \end{aligned}$$

So the grating spacing is 1.64×10^{-6} m.

Question 6.11 Answer

Light of a different colour (i.e. a different wavelength) passes through the same diffraction grating as in Worked example 6.6 (so $d = 1.64 \times 10^{-6}$ m). The first-order diffracted beam is at $\theta_n = 24.1°$. Find the wavelength, λ, of this light.

Appendix A, at the back of this book, considers a further application of trigonometry in science; its use when dealing with vector quantities such as velocity and force. You may find the material useful if you intend to study physics courses in the future.

6.4 Small angle approximations

When the angle under consideration is small, some useful approximations can be employed.

Question

Use your calculator to find $\sin\theta$, $\tan\theta$ and $\cos\theta$ (each to five significant figures) for $\theta = 0.5°$.

Answer

$\sin 0.5° = 8.7265 \times 10^{-3}$, $\tan 0.5° = 8.7269 \times 10^{-3}$ and $\cos 0.5° = 0.99996$.

Question

Convert 0.5° to radians, again giving your answer to five significant figures.

Answer

$$\begin{aligned} 360^\circ &= 2\pi \\ \text{so } 1^\circ &= \frac{2\pi}{360} \\ 0.5^\circ &= 0.5 \times \frac{2\pi}{360} \\ &= 8.7266 \times 10^{-3} \text{ radians} \end{aligned}$$

Comparing the answers to the above questions shows that $\sin\theta \approx \theta$ and $\tan\theta \approx \theta$, when θ is measured in radians, and also that $\cos\theta \approx 1$. These results are true for all small angles, in other words

For all small angles

$$\cos\theta \approx 1$$

For small angles *stated in radians*,

$$\sin\theta \approx \theta \quad \text{and} \quad \tan\theta \approx \theta$$

These ‘small angle approximations’ hold within 0.5% accuracy for angles less that about 0.1 radians (6°). Remember though that the final two approximations are only valid for angles measured in radians.

Small angle approximations arise from the fact that, when θ is small in a triangle like the one shown in Figure 6.27, $h \approx a$ and also the length, b, of the straight side opposite to θ approximates to the length of an arc subtended by θ in a circle with its centre at point P and radius h or a. In other words (on Figure 6.27)

$$b \approx s_h \tag{6.14}$$

$$b \approx s_a \tag{6.15}$$

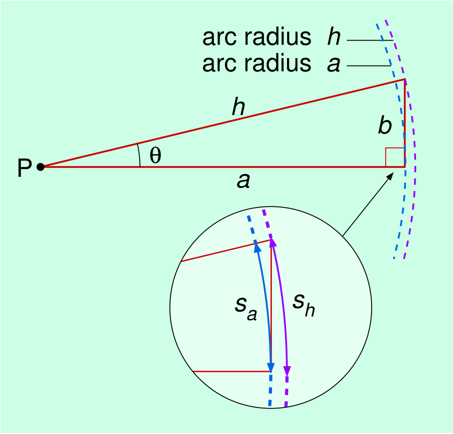


Figure 6.27: A right-angled triangle with a small angle θ

From trigonometry

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{h} \quad \text{and} \quad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{b}{a}$$

For small angle θ we can substitute from Equations 6.14 and 6.15 to give

$$\sin\theta \approx \frac{s_h}{h} \quad \text{and} \quad \tan\theta \approx \frac{s_a}{a}$$

From the definitions of a radian (Equation 6.1)

$$\frac{s_h}{h} = \frac{s_a}{a} = \theta$$

So

$$\sin\theta \approx \theta \quad \text{and} \quad \tan\theta \approx \theta$$

Also,

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{a}{h}$$

so when $h \approx a$ (i.e. for small angles)

$$\cos\theta \approx 1$$

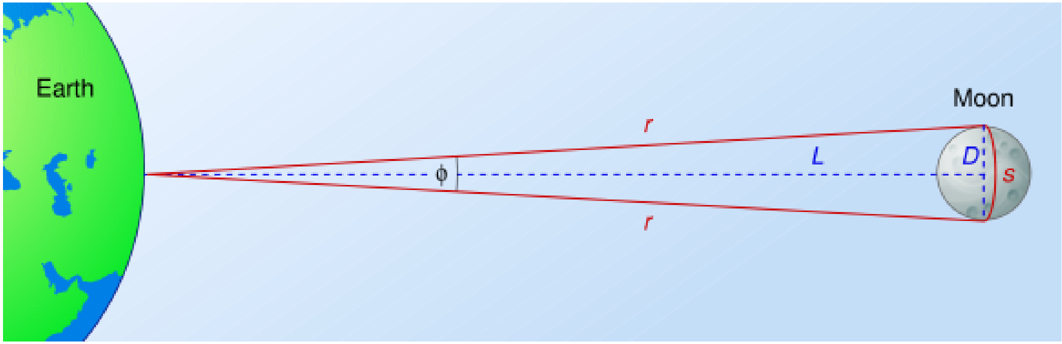


Figure 6.28: Calculating the Moon's diameter (*not to scale*).

Small angle approximations are useful in astronomy, because objects at a great distance subtend a very small angle when observed from the Earth.

An arc drawn from the Earth and encompassing a distant object such as the Moon (see Figure 6.28) has a very similar curvature to a line drawn from one side of the Moon to the other (which is the Moon's diameter) and the distance to the centre of the Moon is approximately equal to the radius of this arc. This gives us a way of calculating the Moon's diameter, the second of the problems raised at the beginning of the chapter. Methodology of the sort illustrated in Worked example 6.7 is frequently used when the size or distance to a distant object is required.

Worked example 6.7

The Moon subtends an angle ϕ of 9.06×10^{-3} radians (from Worked example 6.2) and the distance to the Moon, L, is 3.84×10^{8} mm. Find the Moon's diameter.

Answer

From the definition of the radian (Equation 6.1) and with angles and lengths as shown in Figure 6.28

$$\phi = \frac{s}{r}$$

In this case $s \approx D$ and $r \approx L$ so

$$\phi \approx \frac{D}{L}$$

Reversing this equation and multiplying both sides by L gives

$$\begin{aligned} D &\approx L\,\phi \\ &\approx 3.84\times10^{8}\ \text{m}\times 9.06\times10^{-3} \end{aligned}$$

(remembering from Section 6.1 that strictly speaking, an angle measured in radians can be written without units).

This gives $D \approx 3.48\times10^{6}$ m, i.e. the Moon's diameter is 3.48×10^{6} m.

Question 6.12 Answer

A man standing on a beach observes that a passing car ferry subtends an angle of 3.5°. The ferry is 86 m long. How far is it from the ferry to the man? Assume that the ferry is perpendicular to the direction in which it is being observed, as shown in Figure 6.29.

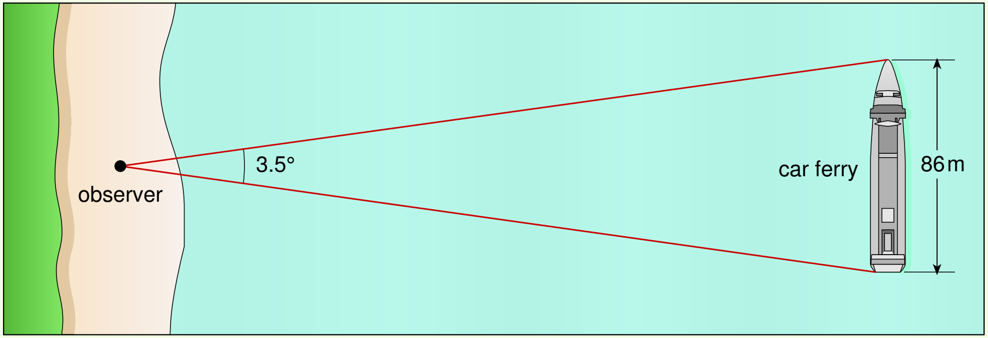


Figure 6.29: A car ferry observed from a beach (*not to scale*).

6.5 Learning outcomes for Chapter 6

After completing your work on this chapter you should be able to:

6.1 demonstrate understanding of the terms emboldened in the text;

6.2 use degrees or radians to measure angles, and convert between these two systems of angular measure;

6.3 find an internal angle in a triangle if you have been told the other two internal angles;

6.4 calculate the length of any side of a right-angled triangle if you have been told the lengths of the other two sides;

6.5 use a scientific calculator to find angles from trigonometric ratios (sin, cos and tan only), and vice versa;

6.6 use trigonometry to find unknown angles and sides in right-angled triangles;

6.7 apply small angle approximations when appropriate;

6.8 apply knowledge gained in this chapter and earlier in the course to scientific examples involving angles and trigonometry.

7 Logarithms

> 'Seeing there is nothing (right well-beloved Students of the Mathematics) that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances.'

Thus wrote John Napier in the preface to his book *Mirifici logarithmorum canonis descripio* in 1614 (the quote is from the English translation of 1616). Napier (1550–1617) was a wealthy Scottish landowner and theologian, who claimed to study mathematics only as a hobby. Despite this, he invented both logarithms (or 'logs' for short) and 'Napier's bones' with the express purpose of making it easier to do multiplications and divisions. Logarithms were in regular use for this purpose well into the second half of the twentieth century.

Nowadays we have electronic calculators and computers to help with long multiplications and divisions, so you may be wondering why this course, written in the twenty-first century, still includes a chapter on logarithms. Over the years, in addition to being an invaluable aid to arithmetic, logarithms have proved themselves to have many applications and they remain widely used in these applications. For example, the pH-scale (used to describe acidity) is based on logarithms, and the curved graph representing the variation of activity with time for a radioactive source (see Chapter 5 Figure 5.34) can be turned into a straight line by plotting the *logarithm* of activity against time. This chapter will explain what logarithms are, and demonstrate some of their uses in modern science.

7.1 Logarithms to base 10

Henry Briggs (1561–1630), the first professor of geometry at Gresham College, London, visited Napier in the summer of 1615 and, with Napier's blessing, developed the type of logarithms known as logarithms to base 10, or 'common logarithms'.

You know, from Section 1.3.1 that, for example,

$$10^6 = 1000\,000$$

$$10^3 = 1000$$

$$10^0 = 1$$

$$10^{-1} = \frac{1}{10} = 0.1$$

$$10^{-5} = \frac{1}{10^5} = 0.000\,01$$

where 10 is known as the base number.

The process of obtaining a logarithm to base 10 (usually described as 'taking the log to base 10') is the *inverse* of raising the base 10 to a power. In each of the above examples the logarithm to base 10 of the number on the right-hand side of the equation is simply the power to which the 10 on the left-hand side is raised. The logarithm to base 10 is abbreviated $\log_{10}$ in this course (you may also see the abbreviation log, without a subscript, used to describe a logarithm to base 10) so, for example,

$10^3 = 1000$ so $\log_{10} 1000 = 3$

10 raised to the power 3 equals 1000

so the logarithm to base 10 of 1000 is 3

$10^0 = 1$ so $\log_{10} 1 = 0$

10 raised to the power 0 equals 1

so the logarithm to base 10 of 1 is 0

$10^{-1} = 0.1$ so $\log_{10} 0.1 = -1$

10 raised to the power –1 equals 0.1

so the logarithm to base 10 of 0.1 is –1

We can say, more generally:

> The logarithm to base 10 of p is the power to which 10 must be raised in order to equal p.
>
> i.e. if $p = 10^n$, then $\log_{10} p = n$.

The definition of a logarithm to base 10 applies for fractional values of n too. For example, you know, from Section 1.3.4 that $\sqrt[3]{10}$ can be written as $10^{1/3}$. This means that

$$10^{1/3} = \sqrt[3]{10} \qquad \text{so} \qquad \log_{10}(\sqrt[3]{10}) = \frac{1}{3}$$

10 raised to the power $\frac{1}{3}$ equals $\sqrt[3]{10}$

so the logarithm to base 10 of $\sqrt[3]{10}$ is $\frac{1}{3}$

In fact, n could be *any* number; you may like to start by using your calculator to check the following to four significant figures (use either the 'x^y' or '∧' button or, if your calculator has one, a button marked '10^x'):

$$10^{0.1235} = 1.329$$

$$10^{3.456} = 2858$$

$$10^{-1.234} = 0.05834$$

From the last of these we can say that

$$10^{-1.234} = 0.05834 \qquad \text{so} \qquad \log_{10} 0.05834 = -1.234$$

10 raised to the power
–1.234 equals 0.05834

so the logarithm to base
10 of 0.05834 is –1.234

Question 7.1

Without further use of a calculator, write down the values of:

(a) $\log_{10} 100$ Answer

(b) $\log_{10} 0.001$ Answer

(c) $\log_{10} \sqrt{10}$ Answer

(d) $\log_{10} 1.329$ Answer

Since taking a logarithm to base 10 is the inverse of raising 10 to a power, the '$\log_{10}$' or 'log' button on a calculator should reverse the operation of the '10^x' button. You can use your calculator to check this for an arbitrarily chosen number, e.g. 4.8; the '10^x' button should give 63 095.734 45 and finding the logarithm to base ten of the latter number returns the display to 4.8.

Question 7.2

Use your calculator to find the following to 4 significant figures:

(a) $\log_{10} 2$ Answer

(b) $\log_{10} 2000$ Answer

Question 7.3

(a) Use your calculator to find $10^{1.5}$ to 4 significant figures. Answer

(b) If $\log_{10} p = 1.5$, what is p? Answer

It is worth noting that

- it is not possible to obtain the logarithm to base 10 of a negative number, or of zero: if you try this on your calculator it will produce an error message.
- it is possible to obtain logarithms of *pure numbers* only; you cannot obtain the logarithm of a quantity possessing units. Strictly, if a quantity possesses units, then it should be divided by those units before taking the logarithm. You will see how this is done in practice in Box 7.1 later in this chapter.

7.2 Logarithmic scales revisited

Logarithmic scales, such as the Richter scale for earthquake magnitude and the decibel scale for relative loudness of sounds, were discussed in Chapter 2. The word ‘logarithmic’ is used to describe such scales simply because they are based on logarithms; both the decibel and the Richter scales are based on logarithms to base 10. It was stated in Chapter 2 that logarithmic scales are used when the quantities being measured vary over a wide range (see Figure 2.2); the answer to Question 7.2 illustrates why logarithms are so useful in this context. The log to base 10 of 2 is 0.3010, but the log to base 10 of 2000, a number a thousand times bigger than 2, is just 3.3010 and it turns out that the log to base 10 of 2000 000 is only 6.3010. Thus taking logarithms gives us a way of coping with a scale that covers a huge range of values.

As a more specific example of this, consider the decibel. This unit was introduced in Chapter 2, but now it can be defined properly. The loudness of a sound in decibels, relative to a threshold value (a sound which is just audible) is given by:

$$\text{relative intensity in decibels} = 10 \times \log_{10}\left(\frac{I}{I_0}\right)$$

where I_0 is the intensity of the threshold sound and I is the intensity of the sound in question. So the sound of a pneumatic drill with an intensity 10^{12} times that of the threshold has:

$$\text{relative intensity in decibels} = 10 \times \log_{10}\left(10^{12}\right) = 10 \times 12 = 120$$

The pH scale, widely used as a measure of acidity, is also based on logarithms to base 10. The pH scale is discussed further in Box 7.1.

Box 7.1 The pH scale

The pH scale was developed by the Danish biochemist Søren Sørenson in 1909. 'pH' is an abbreviation for 'pondis hydrogenii' or 'potential of hydrogen' and the scale is based on a measurement of the concentration of hydrogen ions in the solution in question. Concentration and its units, mol dm^{-3}, were introduced in Box 3.2, and a hydrogen ion is a hydrogen atom which has lost an electron and so is positively charged. The hydrogen ion concentration of pure water at 25 °C is $1 \times 10^{-7}\ \text{mol dm}^{-3}$, whilst that of lemon juice (more acidic than pure water) is about $8 \times 10^{-3}\ \text{mol dm}^{-3}$ and that of household bleach (considerably less acidic than pure water) is about $1 \times 10^{-12}\ \text{mol dm}^{-3}$. Note that the range of values is very wide and also that all of the values are quite small, which makes them rather tricky to deal with. The definition of pH (which handles both of these things) is:

$$\text{pH} = -\log_{10}\left(\frac{\text{hydrogen ion concentration in mol dm}^{-3}}{\text{mol dm}^{-3}}\right)$$

Since the hydrogen ion concentration is measured in units of mol dm^{-3} and we divide by mol dm^{-3} before taking the logarithm, we are obtaining the logarithm of a pure number, as required.

From the definition of pH, the pH of pure water is:

$$-\log_{10}\left(\frac{1\times 10^{-7}\ \mathrm{mol\,dm^{-3}}}{\mathrm{mol\,dm^{-3}}}\right) = -\log_{10}\left(10^{-7}\right) = -(-7) = 7$$

the pH of lemon juice is:

$$-\log_{10}\left(\frac{8\times 10^{-3}\ \mathrm{mol\,dm^{-3}}}{\mathrm{mol\,dm^{-3}}}\right) = -\log_{10}\left(8\times 10^{-3}\right) = -(-2.1) = 2.1$$

and the pH of household bleach is:

$$-\log_{10}\left(\frac{1\times 10^{-12}\ \mathrm{mol\,dm^{-3}}}{\mathrm{mol\,dm^{-3}}}\right) = -\log_{10}\left(10^{-12}\right) = -(-12) = 12$$

Thus we have a much more manageable scale. The entire range of values for hydrogen ion concentration, from $1.0\ \mathrm{mol\,dm^{-3}}$ down to $1\times 10^{-14}\ \mathrm{mol\,dm^{-3}}$, is represented by pH values between 0 and 14. A pH of 7 (the value for pure water) representing a neutral solution, with lower numbers being more acidic and higher numbers being less acidic.

Question 7.4

Calculate, to two significant figures, the pH of the following:

(a) human blood, with a hydrogen ion concentration of 4.0×10^{-8} mol dm^{-3} Answer

(b) hair shampoo, with a hydrogen ion concentration of 3.2×10^{-6} mol dm^{-3}. Answer

7.3 Rules of logarithms

Much of the usefulness of logarithms follows from several rules which are summarized below:

$$\log_{10} 10^n = n \tag{7.1}$$

$$\log_{10}(p \times q) = \log_{10} p + \log_{10} q \tag{7.2}$$

$$\log_{10}\left(\frac{p}{q}\right) = \log_{10} p - \log_{10} q \tag{7.3}$$

$$\log_{10}(p^n) = n \log_{10} p \tag{7.4}$$

Equation 7.1 is a restatement of the definition of a logarithm to base ten. The other rules can be derived from the rules for manipulating exponents, given in Chapter 1. The derivation of Equation 7.2 is given in Box 7.2 for your interest (the derivations of Equations 7.3 and 7.4 are similar).

Box 7.2 Deriving Equation 7.2

From the definition of a logarithm to base 10:

$$\text{If } p = 10^{a} \text{ then } \log_{10} p = a \tag{7.5}$$

$$\text{If } q = 10^{b} \text{ then } \log_{10} q = b \tag{7.6}$$

Multiplying p and q gives:

$$p \times q = 10^{a} \times 10^{b} = 10^{a+b}$$

from the rules for exponents given in Section 1.3.2.

Taking the logarithm to base 10 of both sides:

$$\begin{aligned} \log_{10}(p \times q) &= \log_{10}\left(10^{a+b}\right) \\ &= a + b \quad \text{(from Equation 7.1)} \end{aligned}$$

But $a = \log_{10} p$ from Equation 7.5 and $b = \log_{10} q$ from Equation 7.6 so

$$\log_{10}(p \times q) = \log_{10} p + \log_{10} q$$

We can verify Equations 7.2, 7.3 and 7.4 by substituting numerical values for p, q and n.

If $p = 2$ and $q = 1000$, then from Equation 7.2,

$$\begin{aligned}\log_{10}(2 \times 1000) &= \log_{10} 2 + \log_{10} 1000\\ &= 0.3010 + 3\\ &= 3.3010\end{aligned}$$

To five significant figures, this is the same value as a calculator gives for $\log_{10} 2000$ (as obtained in the answer to Question 7.2), so Equation 7.2 seems reasonable. Note that $\log_{10} 2000$ is *exactly* 3 more than $\log_{10} 2$.

Again using $p = 2$ and $q = 1000$, now in Equation 7.3,

$$\begin{aligned}\log_{10}\left(\frac{2}{1000}\right) &= \log_{10} 2 - \log_{10} 1000\\ &= 0.3010 - 3\\ &= -2.6990\end{aligned}$$

To five significant figures, this is the same value as a calculator gives for $\log_{10} 0.002$, so Equation 7.3 seems reasonable. Note that $\log_{10} 0.002$ is *exactly* 3 less than $\log_{10} 2$.

If $p = 2$ and $n = 3$, then from Equation 7.4,

$$\begin{aligned}\log_{10}\left(2^3\right) &= 3\log_{10} 2\\ &= 3 \times 0.3010\\ &= 0.9030\end{aligned}$$

A calculator gives $\log_{10} 8 = 0.9031$ to four significant figures, almost but not exactly the same as the value obtained for $\log_{10}(2^3)$ by using Equation 7.4. directly. Equation 7.4 seems reasonable. If we had used *exact* values for $\log_{10} 2$ and $\log_{10} 8$ the answers would have been identical, but in working to a limited number of significant figures we need to take care with rounding errors.

Worked example 7.1

Use the fact that $\log_{10} 3 = 0.4771$ to obtain a value for $\log_{10} 3000$ without using a calculator. You should give your answer to four significant figures.

Answer

$$\begin{aligned}\log_{10} 3000 &= \log_{10}(3 \times 1000)\\ &= \log_{10} 3 + \log_{10} 1000 && \text{(from Equation 7.2)}\\ &= 0.4771 + \log_{10} 10^3\\ &= 0.4771 + 3 && \text{(from Equation 7.1)}\\ &= 3.477 \text{ to four significant figures}\end{aligned}$$

Question 7.5

Use the fact that $\log_{10} 3 = 0.4771$, and Equations 7.1 to 7.4 to find the following *without* using a calculator. Give your answers to four significant figures.

(a) $\log_{10} 300$, Answer

(b) $\log_{10} 0.03$, Answer

(c) $\log_{10} 9$. {*Hint*: remember that $9 = 3^2$.} Answer

These rules for the manipulation of logarithms explain how Napier's invention was used to simplify the processes of multiplication and division. Equation 7.2 gives a way of turning multiplication into addition; Equation 7.3 gives a way of turning division into subtraction and Equation 7.4 gives a way of calculating powers and roots. The rules of logarithms have other uses too, as illustrated in Box 7.3 on the next page.

Box 7.3 ***k*-value analysis**

k-value analysis provides a methodology for studying the different factors that affect the size of a biological population. Consider, for example, a population of 24 pairs of owls studied by H. N. Southern in Wytham Wood, near Oxford, in 1952–1953. In order for a pair of owls to have young which themselves will breed, various things must happen: for example, the parents must breed; the eggs must hatch; they must produce fledglings that survive to be owlets; the owlets must live long enough to form pairs. Things can go wrong at every stage! The *k*-value (which you can think of as the ‘killing factor’) is a measure of the killing power of each of the things that can go wrong.

At each stage:

$$k = \log_{10}\left(\frac{N_B}{N_A}\right)$$

where N_B is the number of individuals alive before this stage and N_A is the number of individuals alive afterwards.

For example, 43 eggs were laid (N_2 in Table 7.1) but only 16 eggs hatched (N_3 in Table 7.1) so the *k*-value for this stage is:

$$k_3 = \log_{10}\left(\frac{N_2}{N_3}\right) = \log_{10}\left(\frac{43}{16}\right) = \log_{10}(2.6875) = 0.4293$$

Question

Use the data in Table 7.1 to find $k_5 = \log_{10}\left(\frac{N_4}{N_5}\right)$

Answer

$$\begin{aligned} k_5 &= \log_{10}\left(\frac{N_4}{N_5}\right) \\ &= \log_{10}\left(\frac{15}{9}\right) \\ &= \log_{10}(1.6667) \\ &= 0.2218 \end{aligned}$$

k-value analysis gives an easy way of comparing the effect of different killing factors and k_{total}, the total k-value for all stages is

$$k_{\text{total}} = \log_{10}\left(\frac{N_0}{N_5}\right) = \log_{10}\left(\frac{72}{9}\right) = 0.9031$$

Question

Use the data given in Table 7.1 to find $k_1 + k_2 + k_3 + k_4 + k_5$.

Answer

$$\begin{aligned} k_1 + k_2 + k_3 + k_4 + k_5 &= 0.1498 + 0.0741 + 0.4293 + 0.0280 + 0.2218 \\ &= 0.9030 \end{aligned}$$

Note that, within rounding errors, this is the same as the value calculated for k_{total} from

$$k_{\text{total}} = \log_{10}\left(\frac{N_0}{N_5}\right)$$

The result $k_{\text{total}} = k_1 + k_2 + k_3 + k_4 + k_5$ can also be proved from the rules of logarithms:

$$k_1 = \log_{10}\left(\frac{N_0}{N_1}\right) = \log_{10} N_0 - \log_{10} N_1 \quad \text{(from Equation 7.3)}$$

Similarly

$$k_2 = \log_{10}\left(\frac{N_1}{N_2}\right) = \log_{10} N_1 - \log_{10} N_2$$

and so on, until

$$k_5 = \log_{10}\left(\frac{N_4}{N_5}\right) = \log_{10} N_4 - \log_{10} N_5$$

So

$$\begin{aligned} k_1 + k_2 + k_3 + k_4 + k_5 = (\log_{10} N_0 - \log_{10} N_1) + (\log_{10} N_1 - \log_{10} N_2) + \ldots \\ + (\log_{10} N_4 - \log_{10} N_5) \end{aligned}$$

Apart from $\log_{10} N_0$ and $\log_{10} N_5$, all of the logarithms on the right-hand side are both added and subtracted, so

$$\begin{aligned} k_1 + k_2 + k_3 + k_4 + k_5 &= \log_{10} N_0 - \log_{10} N_5 \\ &= \log_{10}\left(\frac{N_0}{N_5}\right) \\ &= k_{\text{total}} \end{aligned}$$

7.4 Using logarithms to make curves straight

You were introduced, in Chapter 5, to various graphs of different shapes. For example a graph of the area A of a circle against its radius r is a *parabola*; the equation of this graph is $A = \pi r^2$. Similarly, the graph of the number of radioactive nuclei N against elapsed time t is an *exponential*; the equation of this graph is $N = N_0\,e^{-\lambda t}$. Logarithms can be used to turn these and other curved graphs into straight-line graphs, and a knowledge of the rules of logarithms (from Section 7.3) can be used to interpret the resulting straight-line graphs and thus to determine physical constants such as N_0 and λ.

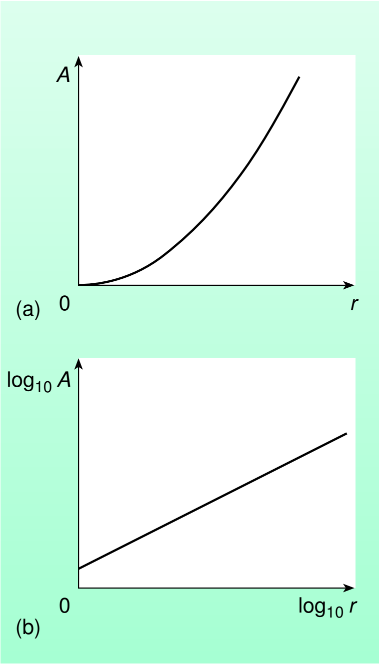


Figure 7.1: Graphs of (a) A against r, and (b) $\log_{10} A$ against $\log_{10} r$ for the equation $A = \pi r^2$.

7.4.1 Log–log graphs

Figure 7.1a shows a graph of A against r for the equation $A = \pi r^2$. One method for turning this curve into a straight line was introduced in Section 5.4. Another method is to plot $\log_{10} A$ against $\log_{10} r$; as shown in Figure 7.1b this also gives a straight line. A graph of this type is known as a 'log–log graph'. But why should it be a straight line?

Taking the log to base 10 of both sides of the equation $A = \pi r^2$ gives:

$$\begin{aligned}\log_{10} A &= \log_{10}\left(\pi r^2\right)\\ &= \log_{10}\pi + \log_{10} r^2 \qquad \text{(from Equation 7.2)}\\ &= \log_{10}\pi + 2\log_{10} r \qquad \text{(from Equation 7.4)}\end{aligned}$$

We can reverse the order of the two terms on the right-hand side to give:

$$\log_{10} A = 2\log_{10} r + \log_{10}\pi$$

This can be compared with the general equation of a straight-line graph, $y = mx + c$ (Chapter 5 Equation 5.1)

$$\begin{array}{cccccc} \log_{10} A & = & 2 & \log_{10} r & + & \log_{10}\pi \\ y & = & m & x & + & c \end{array}$$

gradient

intercept on the vertical axis

This comparison implies that a graph of $\log_{10} A$ against $\log_{10} r$ should be a straight line of gradient 2 and intercept on the vertical axis of $\log_{10}\pi$.

Figure 7.2 is an accurately plotted graph of $\log_{10}(A/\text{cm}^2)$ against $\log_{10}(r/\text{cm})$ for the data in Table 7.2.

Question 7.6 Answer

Find the gradient and intercept on the vertical axis of the straight line shown in Figure 7.2.

r/cm	A/cm^2	$\log_{10}(r/\text{cm})$	$\log_{10}\left(A/\text{cm}^2\right)$
1	π	0	0.497
2	4π	0.301	1.099
3	9π	0.477	1.451
4	16π	0.602	1.701
5	25π	0.699	1.895

Table 7.2: The radius and area of various circles, and corresponding logarithms to base 10

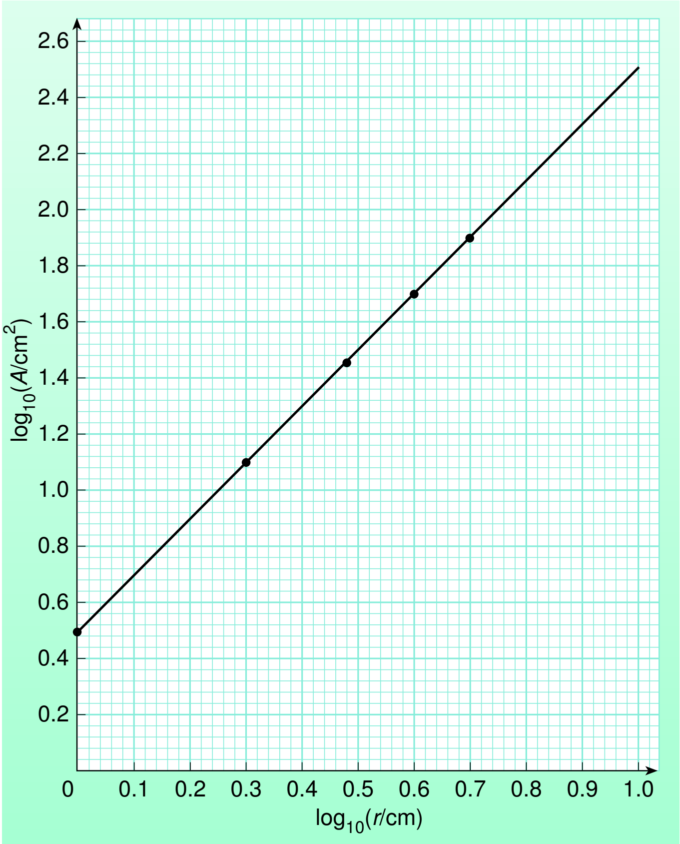


Figure 7.2: A graph of $\log_{10}(A/\text{cm}^2)$ against $\log_{10}(r/\text{cm})$.

Worked example 7.2

If a graph is plotted of $\log_{10} y$ against $\log_{10} x$ for the equation $y = 3x^{-2}$, what will be the gradient and the intercept on the vertical axis?

Answer

Taking the log to base 10 of both sides of the equation $y = 3x^{-2}$ gives

$$\begin{aligned} \log_{10} y &= \log_{10}\left(3x^{-2}\right) \\ &= \log_{10} 3 + \log_{10} x^{-2} \qquad &\text{(from Equation 7.2)} \\ &= \log_{10} 3 - 2\log_{10} x \qquad &\text{(from Equation 7.4)} \end{aligned}$$

We can reverse the order of the two terms on the right-hand side to give

$$\log_{10} y = -2\log_{10} x + \log_{10} 3$$

Comparison with the general equation of a straight-line graph, $y = mx + c$, reveals that $m = -2$ and $c = \log_{10} 3$, so the gradient of the graph will be -2 and the intercept on the vertical axis will be $\log_{10} 3$.

Figure 7.3 shows graphs of y against x and $\log_{10} y$ against $\log_{10} x$ for the equation $y = 3x^{-2}$, but note that it is possible to answer Worked example 7.2 without plotting either of these graphs.

Question 7.7 Answer

If a graph is plotted of $\log_{10} y$ against $\log_{10} x$ for the equation $y = 2x^3$, what will be the gradient and the intercept on the vertical axis?

Plotting graphs of the logarithm of one quantity against the logarithm of another quantity can be used to solve scientific mysteries, as is illustrated in Box 7.4.

Box 7.4 Kepler's third law

The Danish astronomer Tycho Brahe (1546–1601) was a meticulous observer and recorder of data. He developed accurate sighting devices and kept a detailed record of the positions of the planets at regular intervals for more than 20 years. Tycho Brahe's tables provided the data which enabled Johannes Kepler (1571–1630) to work out mathematical relationships describing the motion of the planets.

Plotting the time T it takes for a planet to orbit the Sun (known as its orbital period) against its average distance, a, from the Sun gives a graph of the shape shown in Figure 7.4.

There is clearly a relationship between T and a but what is it? It took Kepler a long time to work this out, but we can use logarithms to help. Let's start by assuming that the relationship is of the form $T = ka^n$ where k and n are constants. The problem now is to find the value of n.

Taking the log to base 10 of both sides of the equation $T = ka^n$ gives:

$$\begin{aligned}\log_{10} T &= \log_{10}(ka^n)\\ &= \log_{10} k + \log_{10} a^n \qquad \text{(from Equation 7.2)}\\ &= \log_{10} k + n\log_{10} a \qquad \text{(from Equation 7.4)}\end{aligned}$$

We can reverse the order of the two terms on the right-hand side to give:

$$\log_{10} T = n\log_{10} a + \log_{10} k$$

Comparison with the general equation of a straight-line graph, $y = mx+c$, shows that a graph of $\log_{10} T$ against $\log_{10} a$ will be a straight line with gradient n and intercept $\log_{10} k$.

Figure 7.5 shows the same data as Figure 7.4, but now the *logarithms* of the variables have been plotted.

Question

What is the gradient of the line in Figure 7.5?

Answer

$$\text{gradient} = \frac{1.5 - 0.0}{1.0 - 0.0} = 1.5$$

This means that $T = ka^{1.5}$, i.e. $T = ka^{3/2}$. Squaring both sides gives $T^2 = k^2a^3$, so the square of the time it takes for a planet to orbit the Sun is proportional to the cube of its average distance from the Sun, i.e. $T^2 \propto a^3$. This is Kepler's third law.

7.4.2 Log–linear graphs

We can turn graphs of equations such as $N = N_0 e^{-\lambda t}$ into straight-line graphs using a similar methodology to the one employed in Section 7.4.1, but now we plot the logarithm of one variable against the other variable itself (not its logarithm). The resultant graph is known as a 'log–linear graph'. Figure 7.6 shows graphs of N against t and $\log_{10} N$ against t for the equation $N = N_0 e^{-\lambda t}$. Note that the graph of N against t is a curve, but that the log–linear graph of $\log_{10} N$ against t is a straight line.

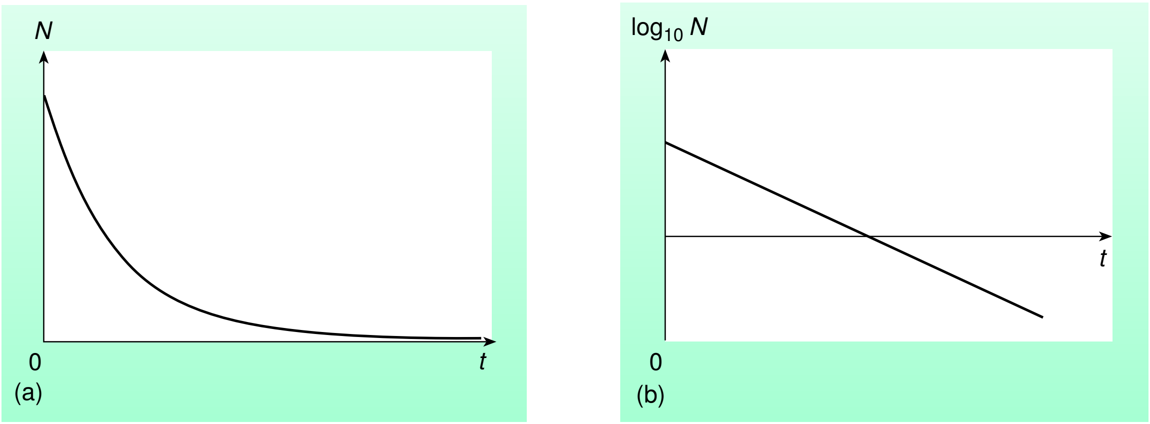


Figure 7.6: Graphs of (a) N against t and (b) $\log_{10} N$ against t for the equation $N = N_0 e^{-\lambda t}$.

To show why the graph of $\log_{10} N$ against t is a straight line we need to start by taking the log to base 10 of both sides of the equation $N = N_0 e^{-\lambda t}$. This gives:

$$\begin{aligned}\log_{10} N &= \log_{10}\left(N_0 e^{-\lambda t}\right)\\ &= \log_{10} N_0 + \log_{10} e^{-\lambda t} \qquad \text{(from Equation 7.2)}\\ &= \log_{10} N_0 - \lambda \log_{10} e \qquad \text{(from Equation 7.4)}\end{aligned}$$

We can reverse the order of the two terms on the right-hand side to give:

$$\begin{aligned}\log_{10} N &= -\lambda t \log_{10} e + \log_{10} N_0\\ &= (-\lambda \log_{10} e)\; t + \log_{10} N_0\end{aligned}$$

This can be compared with the general equation of a straight-line graph, $y = mx + c$

$$\begin{array}{ccccccc} \log_{10} N & = & -\lambda \log_{10} e & t & + & \log_{10} N_0 \\ y & = & \underbrace{m}_{\text{gradient}} & x & + & \underbrace{c}_{\text{intercept on the vertical axis}} \end{array}$$

So a graph of $\log_{10} N$ against t will be a straight line with a gradient of $-\lambda \log_{10} e$ and an intercept on the vertical axis of $\log_{10} N_0$. Note that the gradient of Figure 7.6b is negative, as expected.

Question 7.8 Answer

If a graph is plotted of $\log_{10} n$ against t for the equation $n = n_0\,\mathrm{e}^{at}$ (Chapter 5 Equation 5.3; note that n_0 and a are positive constants), what will be the gradient and intercept on the vertical axis?

Graphs of $\log_{10} y$ against $\log_{10} x$ and $\log_{10} y$ against x are plotted so frequently (though perhaps rather less frequently now than they were in the past, because of computer graph-plotting programs) that special graph paper is available for the purpose. 'Log–linear' (or 'semi-log') graph paper has divisions corresponding to $\log_{10}$ on the vertical axis only, so is useful for plotting graphs of $\log_{10} y$ against x.

Figure 7.7 illustrates the use of log–linear graph paper in investigating the variation of $\log_{10} N$ with t for real experimental data, in this case in an experiment to find the half-life of the excited state of barium-137.

'Log–log graph paper' has divisions corresponding to $\log_{10}$ on both axes, so is useful for plotting graphs of $\log_{10} y$ against $\log_{10} x$.

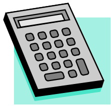
7.5 Logarithms to base e

The previous sections of this chapter have considered logarithms based on powers of 10. It is possible to use numbers other than 10 as the base for logarithms and the other base which is widely used in science is ‘e’, the ‘special number’ introduced in Chapter 5.

In much the same way as taking the logarithm to base 10 is the inverse of raising 10 to a power, so taking a logarithm to base e (abbreviated ln or $\log_e$) is the inverse of raising e to a power.

> The logarithm to base e of p is the power to which e must be raised in order to equal p,
>
> i.e. if $p = e^q$ then $\ln p = q$.

A logarithm to base e is often referred to as a ‘natural logarithm’ and the ‘n’ in the abbreviation ‘ln’ can be thought of as a reminder of this.



Check that you can use your calculator to raise e to various powers. You are likely to be using a button labelled ‘e^x’ in order to do this; the ‘EXP’ button has a totally different use. There is a need to take particular care over the meaning of ‘e’, ‘EXP’ and ‘exp’ since ‘exp’ is sometimes used to mean ‘e to the power’, so $N = N_0e^{-\lambda t}$ is sometimes written as $N = N_0 \exp(-\lambda t)$ and $n = n_0e^{at}$ is sometimes written as $n = n_0 \exp(at)$.

Check that you can get the following results (to four significant figures):

$$e^{3} = 20.09$$
$$e^{0.6931} = 2.000$$
$$e^{-1} = 0.3679$$

Then we can say:

$\mathbf{e^{3} = 20.09}$ so $\mathbf{\ln 20.09 = 3}$

e raised to the power 3 equals 20.09

so the logarithm to base e of 20.09 is 3

$\mathbf{e^{0.6931} = 2}$ so $\mathbf{\ln 2 = 0.6931}$

e raised to the power 0.6931 equals 2

so the logarithm to base e of 2 is 0.6931

$\mathbf{e^{-1} = 0.3679}$ so $\mathbf{\ln 0.3679 = -1}$

e raised to the power –1 equals 0.3679

so the logarithm to base e of 0.3679 is –1

Since taking a logarithm to base e is the inverse of raising e to a power, the ‘ln’ or ‘$\log_e$’ button on a calculator should reverse the operation of the ‘e^x’ button. You can use your calculator to check this for an arbitrarily chosen number, e.g. 1.4; the ‘e^x’ button should give 4.055 199 967 and finding the logarithm to base e of the latter number returns the display to 1.4.

Question 7.9

Use your calculator to find the following to four significant figures:

(a) $\ln 4$, Answer

(b) the number whose natural logarithm is 4. Answer

Note that the rules of logarithms, discussed in Section 7.3, apply to logarithms in *any* base, not just those to base 10. In particular, they apply to logs to base e too, so:

$$\ln e^n = n \tag{7.7}$$

$$\ln (p \times q) = \ln p + \ln q \tag{7.8}$$

$$\ln \left(\frac{p}{q}\right) = \ln p - \ln q \tag{7.9}$$

$$\ln (p^n) = n \ln p \tag{7.10}$$

You may be wondering why logs to base e are useful; why don't we always use logs to base 10? One reason why logs to base e are important stems from the fact that taking a logarithm to base e is the inverse of raising e to a power. This means that equations such as $N = N_0 \mathrm{e}^{-\lambda t}$ can be turned into *simpler* straight-line equations by taking logarithms to base e than is possible by taking logarithms to base 10.

Taking the log to base e of both sides of the equation $N = N_0 \mathrm{e}^{-\lambda t}$ gives:

$$\begin{aligned} \ln N = N &= N_0 \mathrm{e}^{-\lambda t} \\ &= \ln N_0 + \ln \mathrm{e}^{-\lambda t} \qquad \text{(from Equation 7.8)} \\ &= \ln N_0 - \lambda t \qquad \text{(from Equation 7.7)} \end{aligned}$$

We can reverse the order of the two terms on the right-hand side to give:

$$\ln N = -\lambda t + \ln N_0$$

This can be compared with the general equation of a straight-line graph, $y = mx + c$

$$\begin{array}{ccccc} \ln N & = & -\lambda \; t & + & \ln N_0 \\ y & = & \underbrace{m}_{\text{gradient}} \; x & + & \underbrace{c}_{\text{intercept on the vertical axis}} \end{array}$$

So a graph of $\ln N$ against t (see Figure 7.8) will be a straight line with a gradient of $-\lambda$ and an intercept on the vertical axis of $\ln N_0$.

Question

A graph plotted of $\ln N$ against time t for the decay of barium-137 is a straight line of gradient $-4.4 \times 10^{-3}\ \mathrm{s}^{-1}$. What is the decay constant (the constant λ in the equation $N = N_0\, \mathrm{e}^{-\lambda t}$)?

Answer

The gradient of the graph is $-4.4 \times 10^{-3}\ \mathrm{s}^{-1}$ so the decay constant λ is $4.4 \times 10^{-3}\ \mathrm{s}^{-1}$.

Box 7.5 investigates the relationship between decay constant λ and half-life, $t_{1/2}$. The half-life of a radioactive decay process (first introduced in Chapter 5) is the time taken for the number of radioactive nuclei, and hence the activity, to fall by half.

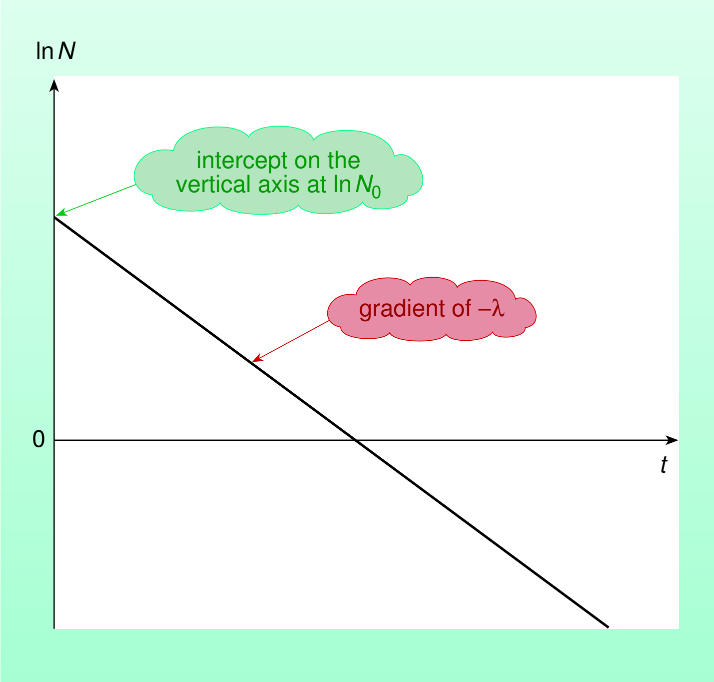


Figure 7.8: A graph of $\ln N$ against t for the equation $N = N_0\, \mathrm{e}^{-\lambda t}$.

Box 7.5 The relationship between decay constant and half-life

The equation $N = N_0\, e^{-\lambda t}$ can be written as:

$$N = N_0 \frac{1}{e^{\lambda t}} \qquad \left(\text{since } e^{-\lambda t} = \frac{1}{e^{\lambda t}}\right)$$

Rearranging gives:

$$e^{\lambda t} = \frac{N_0}{N}$$

At $t = t_{1/2}$, $N = N_0 \times \frac{1}{2}$ (from the definition of half-life in Section 5.4) i.e.

$$\frac{N_0}{N} = 2$$

So $e^{\lambda t_{1/2}} = 2$

Taking the log to base e of both sides of this equation gives:

$$\ln\left(e^{\lambda t_{1/2}}\right) = \ln 2$$

i.e. $\lambda t_{1/2} = \ln 2$ (from Equation 7.7)

$$\lambda = \frac{\ln 2}{t_{1/2}} \qquad \text{or} \qquad t_{1/2} = \frac{\ln 2}{\lambda}$$

Thus a decay constant of $4.4 \times 10^{-3}\ \mathrm{s}^{-1}$ (for barium-137) corresponds to a half-life of $\dfrac{\ln 2}{4.4 \times 10^{-3}\ \mathrm{s}^{-1}} = 1.6 \times 10^{2}$ s to two significant figures.

Question 7.10 Answer

If a graph is plotted of $\ln n$ against t for the equation $n = n_0\,\mathrm{e}^{at}$, what will be the gradient and intercept on the vertical axis? (Note that this is the same equation as used in Question 7.8, but now you are asked to consider a graph of $\ln n$ against t rather than a graph of $\log_{10} n$ against t.)

Box 7.6 The Arrhenius equation

The Arrhenius equation, named after the Swedish chemist Svante Arrhenius (1859–1927), is one of the most important equations in chemistry. It links the rate of a chemical reaction to the temperature at which the reaction takes place. The equation is given by:

$$k_\mathrm{R} = A\,\mathrm{e}^{(-E_\mathrm{a}/RT)} \tag{7.11}$$

where k_R is the 'rate constant' at a particular temperature T, A is the Arrhenius A-factor (or A-factor), E_a is the Arrhenius activation energy (or activation energy) and R is the 'gas constant'.

Taking the log to base e of both sides of Equation 7.11 gives:

$$\begin{aligned}
\ln k_\mathrm{R} &= \ln\left(A\,\mathrm{e}^{(-E_\mathrm{a}/RT)}\right) \\
&= \ln A + \ln \mathrm{e}^{(-E_\mathrm{a}/RT)} \qquad \text{(from Equation 7.8)} \\
&= \ln A - \frac{E_\mathrm{a}}{RT} \qquad \text{(from Equation 7.7)}
\end{aligned}$$

We can reverse the order of the two terms on the right-hand side to give:

$$\begin{aligned}\ln k_{\mathrm{R}} &= -\frac{E_{\mathrm{a}}}{RT} + \ln A \\ &= \frac{-E_{\mathrm{a}}}{R}\,\frac{1}{T} + \ln A\end{aligned}$$

This can be compared with the general equation of a straight-line graph, $y = mx + c$

$$\begin{array}{ccccccc} \ln k_{\mathrm{R}} & = & \dfrac{-E_{\mathrm{a}}}{R} & \dfrac{1}{T} & + & \ln A \\ y & = & m & x & + & c \end{array}$$

gradient

intercept on the vertical axis

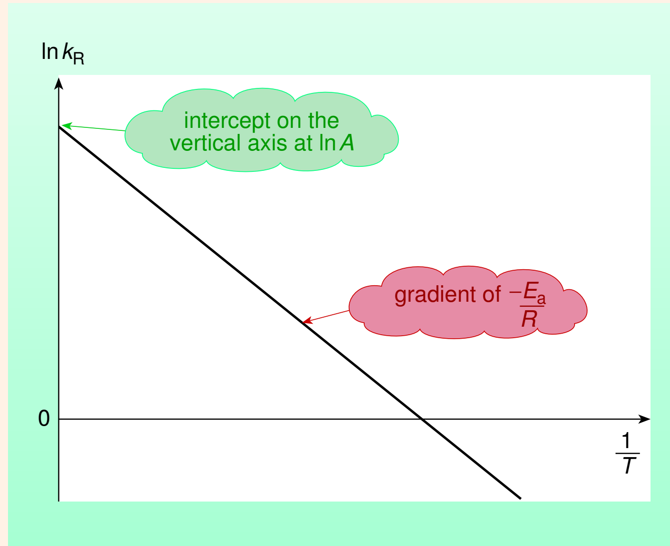


Figure 7.9: A graph of $\ln k_{\mathrm{R}}$ against $1/T$ for the equation $k_{\mathrm{R}} = A\,e^{(-E_{\mathrm{a}}/RT)}$.

So if both A and E_{a} are constants, independent of temperature (a reasonable assumption for most reactions when studied over a limited range of temperature), a graph of $\ln k_{\mathrm{R}}$ against $1/T$ will be a straight line of gradient $-E_{\mathrm{a}}/R$ and intercept on the vertical axis $\ln A$. A graph of $\ln k_{\mathrm{R}}$ against $1/T$ (as shown in Figure 7.9) is referred to as an Arrhenius plot.

The Arrhenius equation accounts remarkably well for the temperature behaviour of the vast majority of chemical reactions, including those which occur in nature. For many living organisms, the temperature of their environment is hugely important, and biological processes are frequently temperature dependent. Biological processes generally involve complex sequences of chemical steps, yet in common with many other composite reactions, they often exhibit an Arrhenius-type behaviour. Figure 7.10 shows an Arrhenius plot for the heartbeat of a diamond-backed terrapin. At lower temperatures, the plot departs from linear behaviour, indicating a different control mechanism.

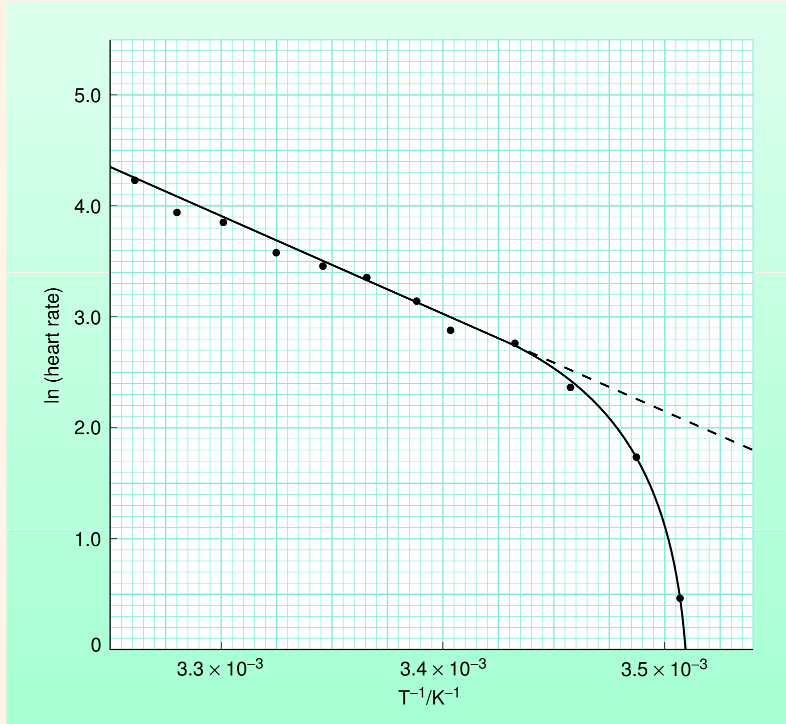


Figure 7.10: An Arrhenius plot of its heartbeat (rate) in the temperature range 18 °C to 34 °C.

7.6 Learning outcomes for Chapter 7

After completing your work on this chapter you should be able to:

7.1 demonstrate understanding of the terms emboldened in the text;

7.2 use a calculator to find the logarithm (to base 10 or base e) of a positive number;

7.3 demonstrate understanding of the relationship between powers of 10 and logarithms to base 10;

7.4 demonstrate understanding of the relationship between powers of e and logarithms to base e;

7.5 use the rules governing the logarithms of products, fractions and powers;

7.6 interpret a graph of $\log_{10} y$ against $\log_{10} x$ for a function of the type $y = a\,x^b$;

7.7 interpret a graph of $\log_{10} y$ against x for a function of the type $y = C\,e^{kx}$;

7.8 interpret a graph of $\ln y$ against x for a function of the type $y = C\,e^{kx}$.

8 Probability and descriptive statistics

Statistical information is a familiar aspect of modern life, which features routinely in, for example, news reports, sports commentaries and advertising. Scientists who have collected large amounts of data by either counting or measuring quantities also rely on statistical techniques to help them make sense of the data. Suppose you had information collected from, say, three thousand patients, all with the same medical condition but undergoing a variety of treatments. First you would need techniques for organizing and describing the data, so that you could present a summary by giving just a few numbers. This is the function of 'descriptive statistics', covered in this chapter. Then you might want to analyse the data in some way, perhaps to decide whether it supports the suggestion that treatment with one particular drug is more effective than other forms of medication. Chapter 9 will look at some of the statistical tests that may be applied to raw data in order to come to objective conclusions about what it really shows.

Statistical techniques offer ways of dealing with variability, and natural variability is something that scientists meet all the time. Each time an experiment or a measurement is repeated, a slightly different result may be obtained; in any group of people there will be a variation in height; the count of background radiation at any individual location fluctuates randomly from moment to moment. It is therefore very important to be able to decide with some measure of certainty whether a particular result could have been obtained simply by chance or whether it has some real significance, and the mathematics of chance and probability underpin all aspects of statistics.

8.1 Chance and probability

> Probability is expectation founded upon partial knowledge. A perfect acquaintance with *all* the circumstances affecting the occurrence of an event would change expectation into certainty, and leave neither room nor demand for a theory of probabilities.'
>
> (George Boole, 1815–1864)

In many branches of science it is not possible to predict with any certainty what the outcome of a particular event will be. There may be several possible outcomes and all the scientist can offer in the way of quantitative prediction is an assessment of the relative likelihood of each of these outcomes. For example, if a man and a woman both carry the cystic fibrosis gene without showing symptoms of the disease, the chances are 1 in 4 that their first child will suffer from the condition. Such assessments of probability are a routine part of genetics, nuclear physics, quantum physics and many other scientific disciplines.

In seeking to understand the nature and rules of probability it is often best to focus initially on everyday examples that are easily visualized. So Sections 8.1.1 to 8.1.4 feature many examples of tossed coins and rolled dice. However, you will also get the opportunity to see how these ideas are applied to some genuine scientific problems: for example, what is the probability that two people planning to have a child will both turn out to be carriers of the cystic fibrosis gene?

8.1.1 Calculating probability

If a process is repeated in identical fashion a very large number of times, the probability of a given outcome is defined as the fraction of the results corresponding to that particular outcome.

$$\text{probability of a given outcome} = \frac{\text{number of times that outcome occurs}}{\text{total number of outcomes}} \tag{8.1}$$

The nature of the fraction in Equation 8.1 shows that the probability of any given outcome cannot be smaller than 0 or larger than 1. A probability of 0 represents impossibility, while a probability of 1 represents inevitability. The closer the probability of a given outcome is to 1, the more likely that outcome is to occur. This is illustrated diagrammatically in Figure 8.1.

When a coin is tossed fairly, the likelihood of it landing on heads is equal to the likelihood of it landing on tails. If it is tossed repeatedly a great many times, it will in theory come up heads half the time: the probability of tossing heads is $\frac{1}{2}$. The theoretical probability of tossing tails is, of course, also $\frac{1}{2}$. The sum of these two probabilities is $\frac{1}{2} + \frac{1}{2} = 1$; i.e. it is certain that when the coin is tossed it will land either on heads or on tails. This is an example of a general rule:

The sum of the probabilities of all possible outcomes is equal to 1.

A probability of 1 represents certainty.

Dice games involve rolling six-sided dice. Each face of a dice is marked with a different score: one, two three, four, five or six. If the dice is not loaded and the rolling is done fairly, then all outcomes are equally likely, so the probability of any one of the six possible outcomes (for example scoring a three) is $\frac{1}{6}$. Again, the sum of the probabilities of all the possible outcomes is $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$.

So on one roll of the dice the probability of scoring a three is $\frac{1}{6}$ and the probability of *not* scoring a three is $\frac{5}{6}$. Another way of expressing this is to say that on a single roll of the dice there is only one way of scoring a three, but there are 5 ways of *not* scoring a three. Clearly, it is more likely than not that a number other than three will be scored. This is just one illustration of another general rule:

The most likely outcome is the one that can occur in the greatest number of ways.

Provided nothing biases the result to make one outcome inherently more likely than others, the definition given by Equation 8.1 can be rewritten to encompass the number of ways in which a particular outcome may come about:

The probability of a given outcome =

$$\frac{\text{number of ways to get that particular outcome}}{\text{total number of possible outcomes}} \tag{8.2}$$

Question

What would be the probability of throwing an odd number on one roll of a dice?

Answer

There are three possible ways of getting an odd number (1, 3 or 5) and six possible outcomes in total, so Equation 8.2 shows that the probability of throwing an odd number is $\frac{3}{6}$, which can be simplified to the equivalent fraction $\frac{1}{2}$.

An alternative way of arriving at this conclusion is to say that as three of the possible outcomes are even and three are odd, the chances of one throw resulting in an odd number are the same as of it resulting in an even number. Hence the probability of an odd number is $\frac{1}{2}$.

Question 8.1

What is the probability of one card drawn at random from a shuffled pack of playing cards being:

(a) a heart, Answer

(b) red, Answer

(c) an ace, Answer

(d) a picture card? Answer

Note: if you are unfamiliar with playing cards, you need the following information. There are 52 cards in a pack, divided into four suits: hearts (red), diamonds (red), spades (black) and clubs (black). Each suit contains 13 cards, made up of one ace, nine 'number' cards (from 2 to 10 inclusive) and three picture cards (Jack, Queen, King).

8.1.2 Probability and common sense

The concept of probability is a purely theoretical one. Strictly speaking, no experiment measures a probability: all that we can measure is the fraction of times a particular outcome occurs in a finite number of attempts. In the *infinitely* long run this fraction is expected to approach the theoretical probability, but in practice we may never attain this limit. You could easily toss a fair coin four times and get four heads. You could even toss it 20 times and still get heads on every single toss, though that would be fairly unlikely. But the more tosses you made the more nearly the fraction $\dfrac{\text{number of heads}}{\text{total number of tosses}}$ would approach its theoretical value of $\frac{1}{2}$.

A failure to appreciate the fact that the number of attempts needs to be *extremely* large before the probability of a particular outcome will reliably approach the theoretical value is at the root of many popular misconceptions about probabilities. One commonly held fallacy about coin tossing is that if the first ten tosses of a coin have produced several more heads than tails, then the eleventh toss is more likely than not to come up tails. This is not true. Although in the extremely long run the imbalance between heads and tails is expected to be negligible, on any one toss heads and tails are equally likely, irrespective of previous history. Coins have no memory!

Question 8.2

(a) You toss a single coin three times. It comes down heads on the first two occasions. What is the probability that you will get heads on the third throw? Answer

(b) If you toss two coins simultaneously and they both come down heads, what is the probability that when you then toss a third coin it will also come down heads? Answer

8.1.3 Expressing probability

According to Equation 8.1, probability is defined as a fraction. However, as you know from Chapter 1, a fraction such as $\frac{1}{4}$ may also be expressed as a decimal number or as a percentage:

$$\frac{1}{4} = 0.25 = 25\%$$

The following statements:

- the probability of event A is $\frac{1}{4}$,
- the probability of A occurring is 0.25,
- there is a 25% probability of the outcome being A,

are therefore all equivalent.

In addition, particularly in spoken language, it is common to say,

- there is a 1 in 4 probability of the outcome being A

and that too is equivalent to the other three statements.

For the rest of this chapter, probabilities will usually be expressed as fractions, but you will meet the other notations in Chapter 9.

8.1.4 Combining probabilities

The probabilities described in Sections 8.1.1 and 8.1.2 related to the outcomes of a single process, such as repeatedly tossing one coin. Now suppose you were to toss three separate coins simultaneously. What is the probability that all three will show heads? One way of tackling this problem is to write all the possible combinations of results. There are in fact eight possible outcomes, all of which are equally likely:

1	H	H	H
2	H	H	T
3	H	T	H
4	H	T	T
5	T	H	H
6	T	H	T
7	T	T	H
8	T	T	T

Of the eight combinations, only one — shown in red — represents the desired outcome of three heads. On the basis of Equation 8.2, the probability of all three coins coming up heads is therefore $\frac{1}{8}$.

The same result can be obtained using the 'multiplication rule for probabilities': the probability that the first coin will show heads is $\frac{1}{2}$, and the same is true for the second and third coins. The probability that all three will show heads is $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}$. Notice carefully how this situation differs from the one featured in Question 8.2: both the scenarios described in Question 8.2 correspond to having a choice only between outcomes 1 and 2 in the list above (because the outcome of the first two tosses is already known as being two heads). The multiplication rule is expressed in its most general form by saying that

> If a number of outcomes occur independently of one another, the probability of them all happening *together* is found by multiplying their individual probabilities.

An example of how this rule applies in a common genetic disease is given in Box 8.1.

Box 8.1 Probability and cystic fibrosis

Cystic fibrosis is the most common genetic disease in white European and American populations. It results from one of a number of mutations (errors) in a single gene that codes for a protein involved in the transport of salts in the cells of the body. A person with cystic fibrosis has numerous symptoms including sticky mucus in the lungs which makes them prone to infections, abnormally salty sweat and problems with the digestion of food. The cystic fibrosis (CF) gene is described as recessive, which means that individuals with only one copy of the gene, so-called 'carriers', show no symptoms of the disease and may be unaware that they carry the gene. Individuals with two copies of the faulty CF gene will show the symptoms of the condition.

Among white Europeans, the probability of being a carrier is $\frac{1}{25}$.

For a child whose parents are both carriers, the probability of inheriting a copy of the CF gene from both parents is $\frac{1}{4}$. This is therefore the probability that the child of such parents will have symptoms of the disease.

Question

Assuming that the gene is equally likely to be carried by men and women, what is the probability that any two people planning to have a child together would both be carriers?

Answer

The probability of both partners being carriers is $\frac{1}{25} \times \frac{1}{25} = \frac{1}{625}$.

Question

What is the probability of a child born to white European parents having cystic fibrosis?

Answer

The probability that both parents are carriers is $\frac{1}{625}$, and the probability that a child whose parents are both carriers will have the disease is $\frac{1}{4}$. So the probability of a child born to white European parents having cystic fibrosis is $\frac{1}{625} \times \frac{1}{4} = \frac{1}{2500}$

(In fact the figure quoted for babies born with cystic fibrosis in the UK is about 1 in 2000, somewhat higher than this calculation would suggest.)

Question 8.3

(a) If you toss two coins at the same time, what is the probability of getting two tails? Answer

(b) If you throw a pair of dice, what is the probability of getting a pair of sixes? Answer

Question 8.4

Under identical conditions, a seed of each of three different species of plant A, B, and C, has a germination probability of $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$, respectively. If we have one of each type of seed, what is the probability that:

(a) the seed of A and the seed of B will both germinate? Answer

(b) one seed of each of the three species will germinate? Answer

(c) no seed of any of the species will germinate? (*Hint*: first work out the probability of non-germination for each type of seed individually.) Answer

Another situation in which you might need to combine probabilities occurs when outcomes are mutually exclusive (i.e. cannot occur together). For example, what is the probability of getting *either* a three *or* a five on a single roll of a die? One way of working this out is to say that there are six possible outcomes altogether and two of them correspond to the desired outcome. So from Equation 8.2, the probability of the desired outcome is $\frac{2}{6} = \frac{1}{3}$. The same result can be obtained using the 'addition rule for probabilities'. The probability of throwing a three is $\frac{1}{6}$ and the probability of throwing a five is also $\frac{1}{6}$, so the probability of throwing *either* a three *or* a five is $\frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$. Again, this example illustrates a general rule:

If several possible outcomes are mutually exclusive, the probability of *one or other* of these outcomes occurring is found by adding their individual probabilities.

Worked example 8.1

One card is drawn from a shuffled pack of 52. What is the probability of the card being either a heart or a diamond? (For a description of a standard pack of cards, see the comment with Question 8.1.)

The card cannot be both a heart and a diamond: these outcomes are mutually exclusive.

The probability of the card being a heart is $\frac{1}{4}$.

The probability of the card being a diamond is $\frac{1}{4}$.

So the probability of the card being either a heart or a diamond is:

$$\frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

{Note: Since both diamonds and hearts are red suits, the question is equivalent to asking ‘what is the probability of a single card drawn from the pack being red?’ This was posed as Question 8.1b and answered then by a different route, though of course the result was the same!}

Question 8.5 Answer

If you were to draw one playing card from a pack of 52, what would be the probability of that card being either the Jack, Queen or King of diamonds?

There are also cases in which *both* the addition and multiplication rules operate. For example:

Question

What is the chance that in a family of three children only one will be a boy?

Answer

Assuming that the sex of a child is independent of the sexes of its siblings, the probability that the first child is a boy is $\frac{1}{2}$, the probability that the second is a girl is $\frac{1}{2}$, and the probability that the third is also a girl is $\frac{1}{2}$. So the probability of this particular combination (boy–girl–girl) is

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

But in a family with just one boy and two girls, the boy may be the eldest, the middle or the youngest child, and these possibilities are mutually exclusive. So the probability of the family consisting of a boy and two girls (born in any order) is

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

(Note that in fact the assumption that a baby is just as likely to be a boy as a girl is not quite true. UK statistics show that for every 100 girls born, 106 boys are born.)

As with the coin-tossing example earlier, you may find that a table of the possibilities helps in visualizing the situation. Of the eight possible combinations of three children, only three — shown in red — comprise one boy and two girls.

First child	Second child	Third child	Boys	Girls
Boy	Boy	Boy	3	0
Boy	Boy	Girl	2	1
Boy	Girl	Boy	2	1
Boy	Girl	Girl	1	2
Girl	Boy	Boy	2	1
Girl	Boy	Girl	1	2
Girl	Girl	Boy	1	2
Girl	Girl	Girl	0	3

Question 8.6 Answer

If you toss two coins simultaneously, what is the probability of getting one head and one tail?

8.1.5 Probability ratios

Probability calculations are important in many branches of science, but nowhere more so than in genetics. Box 8.2 describes early work in the field and provides some illustrative data, based on plant-breeding experiments.

Box 8.2 Mendel's peas

Gregor Mendel (1822–1884) was an Austrian monk whose experiments in breeding the garden pea laid the foundations of the science we now call genetics. Mendel did not know about genes in the way that they are understood today, still less about chromosomes and DNA. The rules of inheritance he developed were based on what he observed of the external characteristics of his plants, and the probabilities of plants with particular characteristics arising from specific breeding crosses carried out in the following way:

1. Mendel used pollen from a plant to fertilize the flowers of the same plant (so called 'self-pollination') for several generations until he was sure he had 'pure breeding' plants, i.e. plants that always produced offspring identical in appearance to themselves. He called these pure-breeding plants the P-generation ('P' for parental).

2. He then took pollen from one P-generation plant and used it to fertilize another P-generation plant with a different characteristic. By this process of 'cross-pollination' a pure-breeding purple-flowered variety could be crossed

with, for example, a pure-breeding white-flowered one. Mendel called the offspring of this cross the F_1 (first filial) generation.

3. Finally members of the F_1-generation were self-pollinated and the offspring of this process were called the F_2 (second filial) generation.

Mendel investigated seven pairs of contrasting characteristics of his pea plants. His results relating to three of these pairs of characteristics — flower colour, seed shape and stem length, are shown below. Mendel found these characteristics to be independent: the fact that a particular plant had white flowers had no bearing on whether its seeds were round or wrinkled or on what height the plant was.

Flower colour:	P (purple) crossed with P (white)
	F_1 all purple-flowered
	F_2 705 purple- and 224 white-flowered
Seed shape:	P (round) crossed with P (wrinkled)
	F_1 all seeds round
	F_2 651 seeds round and 207 seeds wrinkled
Stem length:	P (tall) crossed with P (short)
	F_1 all plants tall
	F_2 787 tall plants and 277 short plants

Note that in the case of Mendel's peas, the heights of the plants were not distributed across a continuous range: there was no difficulty in deciding whether a particular plant was 'tall' or 'short'.

Before working with this data, it is important to understand how the results have been presented. Raw data from breeding experiments come in terms of descriptions and numbers, as with the examples given in Box 8.2, but results are often reported by expressing the numbers in the form of a ratio. For example, in the F_2-generation, Mendel obtained 705 plants with purple flowers and 224 with white flowers. Another way of expressing this is to say that purple- and white-flowered plants appeared in the ratio 705 : 224 (said as '705 to 224').

We can think of ratios as simply another way of writing fractions. If, for instance, we discovered from a paint chart that a green paint had been mixed from yellow paint and blue paint in the ratio 3 : 2, we would understand that the green paint was made up of three parts yellow paint and two parts blue paint. In other words, $\frac{3}{5}$ of the mixture was yellow and $\frac{2}{5}$ was blue. Adding both sides of the ratio together has given us the denominator of the fractions. Knowing the denominator, it is then easy to express the ratio in terms of percentages: $\frac{3}{5} = \frac{60}{100}$ so 60% of the mixture is yellow and 40% is blue. A 60 : 40 ratio is exactly the same as a 3 : 2 ratio — it is just a matter of multiplying or dividing both sides of the ratio by 20. Sometimes it is convenient to simplify even further, in this case by dividing both sides by two to express the 3 : 2 ratio in the equivalent form of 1.5 : 1. Note that, like fractions, ratios do not have units attached to them.

Ratios are quoted in many applications. For example, fertilizers are characterized on their labelling by the ratio of two or three major ingredients, each indicated by a letter. These letters are N (for nitrogen, which is required for leaf growth), P (for phosphorus, which in the form of phosphates is required for root development) and K (for potassium, which in the form of potash is required for flowers and fruit). Typical ratios for three common types of fertilizer are shown in Table 8.1.

	N	P	K	Others
bone meal	1	5	0	19
lawn tonic	4	1	0	5
tomato food	6	5	9	80

Table 8.1: Ratios of ingredients in common fertilizers expressed as ratios N : P : K : others

Question

What is the fraction of P in bone meal?

Answer

The fraction of P in the whole is $\dfrac{5}{1+5+19} = \dfrac{5}{25} = \dfrac{1}{5}$.

Question

What is the percentage of N in lawn tonic?

Answer

The fraction of N in the whole is $\dfrac{4}{4+1+5} = \dfrac{4}{10} = \dfrac{40}{100}$.

So lawn tonic contains 40% N.

Question

What is the percentage of K in tomato food?

Answer

The fraction of K in the whole is $\dfrac{9}{6+5+9+80} = \dfrac{9}{100}$.

So tomato food contains 9% K.

As already noted for the paint example, it is quite common for ratios to be expressed in a form such that one of the parts is 1, even if this means that the other part is a decimal number. Question 8.7 gives an illustration of a ratio expressed in such a way.

Question 8.7 Answer

In the atmosphere, the ratio of the volume of oxygen to the volume of other gases is 0.26 : 1. What percentage of the atmosphere is oxygen?

The ratio of 705 : 224 that Mendel obtained for purple- to white-flowered plants (see Box 8.2) can be simplified by dividing both sides of the ratio by 224 to obtain the equivalent ratio of 3.15 : 1. Notice that one side of this ratio is exact: $\frac{224}{224}$ is *exactly* equal to 1. However, the other side is not exact and a choice has to be made about how many significant figures to quote; 2 or 3 significant figures are usually sufficient in this context. His data relating to the other independent pairs of characteristics involving seeds and stem lengths can be simplified in a similar way by dividing the larger number by the smaller, to obtain:

flowers	purple : white	=	705 : 224	=	3.15 : 1
seeds	round : wrinkled	=	651 : 207	=	3.14 : 1
stems	tall : short	=	787 : 277	=	2.84 : 1

In each case the ratio is close to 3 : 1. In other words, the character from the P-generation that was present in all members of the F_1-generation is present in only about $\frac{3}{4}$ of the F_2-generation. By the same token, the character that completely vanished in the F_1-generation reappears in about $\frac{1}{4}$ of the F_2-generation. In fact, modern understanding of genetics leads to the theoretical prediction of a 3 : 1 ratio; the slight deviations observed in experiments like Mendel's are the same as those observed when tossing a coin. The more tosses of the coin, the more nearly the ratio of heads : tails approaches 1 : 1. Similarly, the more pea plants included in the experiments, the more nearly the ratios would be expected to approach 3 : 1.

The examples of Mendel's experiments on peas concerned the inheritance of just a single pair of alternative characteristics: flowers were either purple or they were white; seeds were either round or they were wrinkled; stems were either tall or they were short. When there are more than two options for particular characteristics, the calculations become a little more complicated, but the principles remain exactly the same, as demonstrated by the following worked example.

Question

On a maize cob, four types of grain can be distinguished: dark smooth ones, dark wrinkled ones, pale smooth ones and pale wrinkled ones. The aggregate results of counting numbers of the four types on 20 cobs all from the same plant were:

dark smooth	dark wrinkled	pale smooth	pale wrinkled
4791	1587	1617	531

Assuming that the theoretical ratios for these characteristics are whole numbers, what would be the theoretical probability that a single grain chosen at random from a large number of cobs would be a pale smooth one?

Answer

Dividing through by the smallest number in the sample, which in this case is 531, gives:

dark smooth	dark wrinkled	pale smooth	pale wrinkled
9.02	2.97	3.05	1

If it is assumed that the theoretical ratios are whole numbers, these data strongly suggest that the ratios would be:

dark smooth	dark wrinkled	pale smooth	pale wrinkled
9	3	3	1

The theoretical fraction of grains that are pale and smooth is therefore

$$\frac{3}{9+3+3+1} = \frac{3}{16}$$

This is also the probability of one grain selected at random being pale and smooth. This probability could be expressed as a fraction $\left(\frac{3}{16}\right)$, a decimal number (0.1875) or a percentage (18.75%).

8.2 Descriptive statistics

Scientists collect many different types of information, but sets of data may be very loosely classified into two different types. In the first type, so-called 'repeated measurement', an individual quantity is measured a number of times. An astronomer wanting to determine the light output of a star would take many measurements on a number of different nights to even out the effects of the various possible fluctuations in the atmosphere that are a cause of stars 'twinkling'. In the second type of investigation, so-called 'sampling', a proportion of the members of a large group are measured or counted. A botanist interested in the average size of Primrose plants in a wood would try to choose representative samples of plants from different parts of the wood and measure those.

8.2.1 Repeated measurements

Scientists are always concerned with the reliability and precision of their data, and this is the prime reason for them to repeat measurements many times. Consider the photograph in Figure 8.2, which was produced by a diffraction grating illuminated with red light (see Box 6.6). To determine the wavelength of the light it would be necessary to measure the distances between the lines. Because the lines are rather fuzzy each measurement would need to be repeated a number of times.

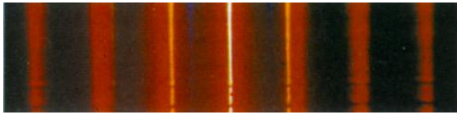


Figure 8.2: The pattern formed by a diffraction grating

Generally the process of repeating measurements of a particular quantity would lead to a number of slightly different results being obtained. Measured values of one quantity that are scattered over a limited range like this are said to be subject to 'random uncertainty'. Measurements for which the random uncertainty is small (i.e. for which the range over which the measurements are scattered is small) are described as precise. The 'best' estimate scientists could make of the distance between each line in Figure 8.2 would be some sort of average of the measured values.

The scatter inevitably associated with raw data begs various questions. For instance,

- how close to the 'true' value is this calculated average value?
- how close to the 'true' value is one typical measurement likely to be?
- conversely, how probable is it than any given measurement will be close to the average value?

Matters will be further complicated if there is some inherent error or bias in the measuring instrument, such that all the readings are, say, too large by a fixed amount. Such measurements are said to have a 'systematic uncertainty'. Note that unless measuring instruments can be constantly checked against one another, it is easy for quite large systematic uncertainties to creep unnoticed into measurements. Measurements for which the systematic uncertainty is small are described as accurate. Of course to get anywhere near to the 'true' value of a quantity, measurements have to be both accurate and precise!

8.2.2 The distribution of repeated measurements

As noted in the previous section, if the same quantity is measured repeatedly, the results will generally be scattered across a range of values. This is perhaps best illustrated using a real example. Table 8.2 shows 10 measurements of a quantity called the 'unit cell constant' for an industrial catalyst used in the refining of petrol; this is an important quantity which determines how well the catalyst works, and can be measured by X-ray diffraction techniques. Notice that the cell constant is very small and is measured in nanometres.

Measurement	Cell constant/nm
1	2.458
2	2.452
3	2.454
4	2.452
5	2.459
6	2.455
7	2.464
8	2.453
9	2.449
10	2.448

Table 8.2: Repeated measurements of the unit cell constant for a batch of industrial catalyst

It is always difficult to see patterns in lists or tables of numbers. If the data is put into the form of a histogram, as has been done in Figure 8.3, the task becomes much easier. The histogram provides a visual representation of the way in which the measurements are distributed across a range of values. In fact the pattern on Figure 8.3 is not particularly obvious, because the data set is quite small, consisting of only ten measurements.

When the number of measurements is increased, the variation in the height of the bars gradually becomes smoother, as illustrated in Figure 8.4.

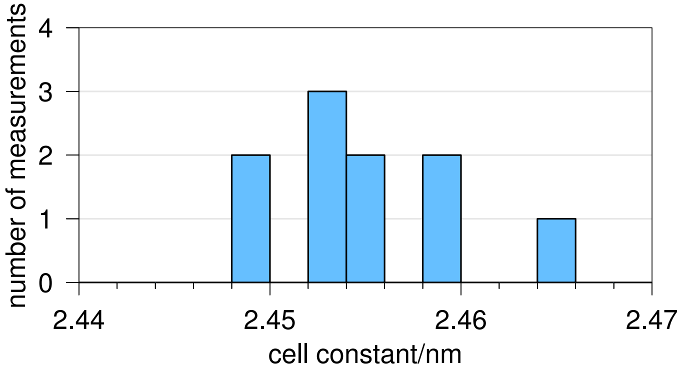


Figure 8.3: Histogram of data from Table 8.2.

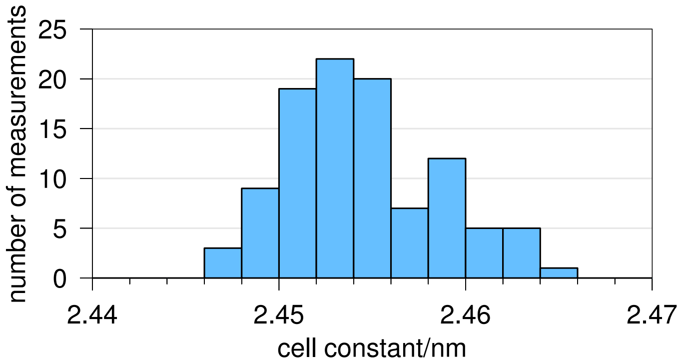


Figure 8.4: Distribution of a larger number of repeated measurements.

When substantially more measurements have been accumulated, the size of the intervals can be reduced while still having a reasonable number of measurements within each interval. This again tends to produce a smoother distribution, as shown in Figure 8.5. Note the changes of vertical scale between Figure 8.3, 8.4 and 8.5.

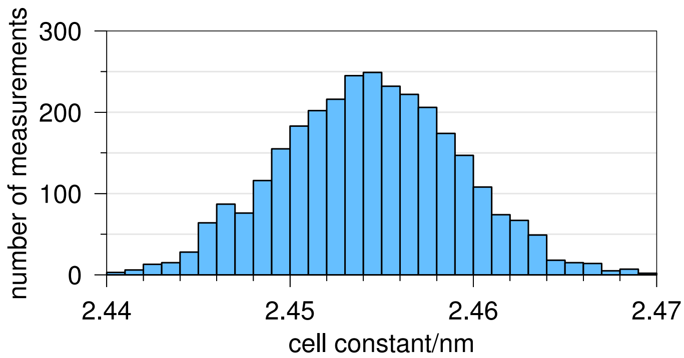


Figure 8.5: The distribution becomes smoother as the number of measurements increases.

With an extremely large number of measurements *and* very small intervals on the horizontal axis, the 'envelope' of the distribution will tend to become a smooth bell-shaped curve, like that in Figure 8.6.

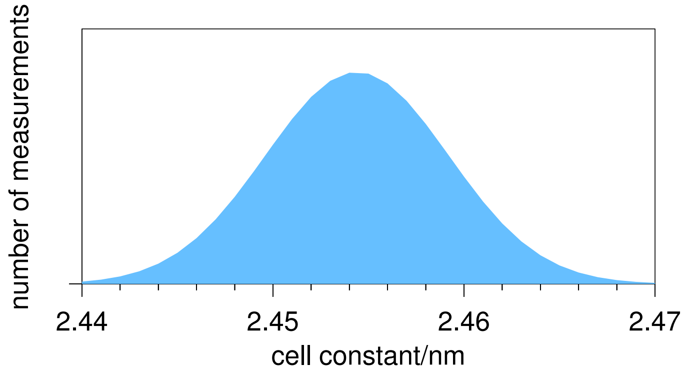


Figure 8.6: The distribution for an extremely large number of measurements.

These distributions all give some impression of the spread of the measurements, and the way the results cluster at the peak of the distribution in Figure 8.6 suggests that this peak might represent the average or 'best estimate' value. However, a scientist would want a more quantitative and succinct way to describe such results and to communicate them to other people working on similar problems. The mean and standard deviation are the measures most commonly used to summarize large sets of data with just a few numbers.

8.2.3 Mean and standard deviation for repeated measurements

In everyday terms, everybody is familiar with the word ‘average’, but in science and statistics there are actually several different kinds of average used for different purposes. In the kind of situation exemplified by Table 8.2, the sort to use is the mean (or more strictly the ‘arithmetic mean’) For a set of measurements, this is defined as the sum of all the measurements divided by the total number of measurements made.

Question

What is the mean of the results in Table 8.2?

Answer

The sum of all the measurements is 24.544 nm. There are 10 results, so the mean value is $\frac{24.544\ \text{nm}}{10}$, or 2.4544 nm to 5 significant figures. (The reason for giving the result to this number of significant figures will be discussed shortly, but for the moment let us proceed without worrying too much about this aspect of the calculation.)

To turn this description of how to calculate a mean into a formula, each element has be allocated a symbol. So let us say that we have made n measurements of a quantity x. Then we can call the individual measurements $x_1, x_2, x_3, \ldots x_n$ (where x_1 is properly said either as ‘x subscript one’ or as ‘x sub one’, but also sometimes as ‘x one’ provided the meaning remains clear). The mean value of any quantity is usually denoted by writing a bar over the quantity so the mean of x is written as $\overline{x}$ (and said ‘x bar’). Possible (and correct) formulae are therefore:

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

or

$$\overline{x} = \frac{1}{n}(x_1 + x_2 + x_3 + \ldots + x_n)$$

However, the sum is tedious to write out, so a special ‘summation’ sign, $\sum$ (the Greek capital letter sigma), is used to denote the adding up process, and the mean of n measurements can be neatly written as:

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad (8.3)$$

The $i = 1$ below the summation sign indicates that the first value for x_i in the sum is x_1, and the n above it indicates that the last value in the sum is x_n. In other words, all integer values of i (x_1, x_2, x_3, etc.) are to be included up to x_n. (The summation sign with the information attached to it is usually said as ‘sum of x sub i from one to n’.)

We now want a quantitative way of describing the spread of measurements, i.e. the extent to which the measurements ‘deviate’ from the mean. There are 5 steps required to do this, which are laid out below, and Table 8.3 shows the results of following this ‘recipe’ for the data in Table 8.2:

Step 1

Calculate the deviation of each measurement. The deviation d_i of any individual measurement is defined as the difference between that measurement and the mean of the set of measurements:

$$d_i = x_i - \overline{x} \tag{8.4}$$

Notice that the value of d_i may be positive or negative depending on whether a particular measurement is larger or smaller than the mean of the set of measurements. At this stage the deviations have been expressed as decimal numbers.

Step 2

Calculate the squares of each of the deviations (i.e. d_i^2). These will, of course, all have positive values.

By this stage the values have become very small so the column has been headed in such way that the numbers entered in the column represent the value of d_i^2 divided by 10^{-5}.

Step 3

Add together all the squares of the deviations (i.e. $\sum_{i=1}^{n} d_i^2$).

Step 4

Divide by the total number of measurements (i.e. n) to obtain the mean of all the square deviations. This may be written as: Table 8.3

$$\overline{d_i^2} = \frac{1}{n}\sum_{i=1}^{n} d_i^2 \tag{8.5}$$

Step 5

Take the square root of this mean to obtain the 'root mean square deviation' s_n. It is this quantity s_n that is known as the standard deviation. Step 5 may be written as:

$$s_n = \sqrt{\overline{d_i^2}}$$

or, substituting for $\overline{d_i^2}$ from Equation 8.5, as:

$$s_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2} \tag{8.6}$$

Since d_i was defined in Equation 8.4 as $(x_i - \overline{x})$, one final substitution into Equation 8.6 gives s_n in its most frequently used format:

The standard deviation s_n for n repeated measurements of the same quantity x is given by

$$s_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2} \tag{8.7}$$

At the end of this process, we can summarize all the data in Table 8.2 just by saying that the ten measurements had a mean of 2.4544 nm and standard deviation 0.0046 nm. The calculation of standard deviation is given in Table 8.3.

There are several things worth noting about this result and the data in Table 8.3.

First, all the quantities have units associated with them. The values of x_i were measured in nanometres, so deviations will also be in nanometres and the squares of the deviations in nm^2, as shown in the column headings in the table.

A second useful feature to notice is that the sum of all the deviations is equal to zero.

$$\sum_{i=1}^{n} d_i = 0$$

If you are interested in knowing why this is always true, there is an explanation in Box 8.4 (though you do not need to work through the full explanation in order to make use of the result). At the end of Step 1 it is well worth adding up all the values you have calculated for the deviations to ensure that they do indeed total zero. If they don't, you have made an arithmetic slip somewhere which needs to be put right before you proceed to Step 2.

Box 8.4 The sum of the deviations is always equal to zero

It is quite easy to work out from first principles the reason for the sum of the deviations being zero in the special case in which the set consists of just two measurements, x_1 and x_2. The mean would then be:

$$\overline{x} = \frac{x_1 + x_2}{2}$$

$$\begin{aligned}
\text{so} \quad d_1 &= x_1 - \overline{x} = x_1 - \frac{x_1 + x_2}{2} = x_1 - \frac{x_1}{2} - \frac{x_2}{2} \\
\text{and} \quad d_2 &= x_2 - \overline{x} = x_2 - \frac{x_1 + x_2}{2} = x_2 - \frac{x_1}{2} - \frac{x_2}{2}
\end{aligned}$$

Therefore

$$\begin{aligned}
d_1 + d_2 &= \left(x_1 - \frac{x_1}{2} - \frac{x_2}{2}\right) + \left(x_2 - \frac{x_1}{2} - \frac{x_2}{2}\right) \\
&= \left(x_1 - \frac{x_1}{2} - \frac{x_1}{2}\right) + \left(x_2 - \frac{x_2}{2} - \frac{x_2}{2}\right) \quad \text{(rearranging the terms)} \\
&= 0
\end{aligned}$$

This argument can be extended to any number of values of x; as an exercise in algebra you might like to try it for three measurements. However many values of x are chosen, it is always the case that the sum of the deviations is zero.

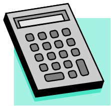
Looking now at the details of the calculation, the original measurements of length were made to the nearest picometre (i.e. 0.001 nm), represented by 3 places of decimals (i.e. 3 digits after the decimal point). More digits were carried in the calculations to avoid rounding errors. However, what is the appropriate number of digits to quote in the final answer? Well, when we added up all the 10 results in Table 8.3, we obtained $\sum_{i=1}^{n} x_i = 24.544$ nm (i.e. 5 digits in total). We divided this sum by an exact number (10) so we are entitled to retain 5 digits in the result of this division, giving $\overline{x}$ as 2.4544 nm. It is therefore valid to retain one more decimal place in the mean value than we had in each of the measurements individually. After all the whole point of repeating the measurement many times and averaging is to improve our confidence in our final result! Having quoted the mean as $\overline{x} = 2.4544$ nm, it then makes sense to quote the standard deviation as 0.0046 nm.

The fact that here the standard deviation is quite small in comparison to the mean shows why, in this context, it is more sensible to think in terms of places of decimals rather than significant figures. Because leading zeroes do not count as significant, the standard deviation is actually only given to 2 significant figures, whereas the mean is given to 5. In such circumstances it is easier to think of the mean and the standard deviation as being expressed to the same number of decimal places (always assuming of course that they are given the same units).

In summary, it is often reasonable to give the mean to one more decimal place (or one more significant figure) than was used for each of the individual measurements, and then to quote the standard deviation to the same number of decimal places as the mean.

8.2.4 Using a calculator for statistical calculations

Table 8.3 shows all the values for each step in the process of calculating a standard deviation, so that you can see what the operations encapsulated by Equation 8.7 actually entail, but you will probably be relieved to hear that it is not usually necessary to carry out such detailed calculations. Scientific and graphics calculators (or computer spreadsheets) can do most of the drudgery for you.



You will need to consult the instructions for your own calculator in order to find out how to do this, but usually the process involves the following steps.

Step 1

Put the calculator into statistical mode.

Step 2

You should then be able to input all the data; sometimes the data is stored via a memory button, in other cases it can be entered and displayed as a list. Try this out with the following set of numbers:

8, 6, 9, 12, 10

Step 3

Having input the data, you can then get most calculators to tell you the number of items of data. If your calculator can do this, it should return the answer ‘5’ here. It doesn’t matter if your calculator doesn’t have this function, but if it does it’s well worth using this checking device. If you have to input a long string of data values, it’s quite easy to miss one out inadvertently!

Step 4

When you know you have the data correctly stored, find out how to display the mean; you should get the answer '9' here.

Step 5

Now find out how to display the standard deviation. Many calculators use the symbol σ_n for standard deviation, rather than s_n (σ is the lower case version of the Greek letter sigma). Do be careful with this step: your calculator may also have a button labelled σ_{n-1} or s_{n-1}. Don't use it by mistake! You should get the answer '2' here.

Once you are sure you know how to use your calculator to perform calculations of mean and standard deviation, apply this skill to Question 8.8. To answer such questions, you could choose to work out a full table similar to Table 8.3, but that it is a very time-consuming process, so it is worth becoming confident in using the statistics buttons on your calculator.

Question 8.8 Answer

A sample of a particular manufacturer's 'coarse round wire' was measured at ten points along its length. The data is given in Table 8.4.

Calculate the mean and standard deviation of these measurements.

Measurement	Diameter/mm
1	1.09
2	1.00
3	1.25
4	1.24
5	1.29
6	0.89
7	1.09
8	1.14
9	1.22
10	1.01

Table 8.4: Repeated measurements of the diameter of a wire.

8.2.5 How likely are particular results?

In real experiments, as opposed to hypothetical ones, it is very rare that scientists make a sufficiently large number of measurements to obtain a smooth continuous distribution like that shown in Figure 8.6. However, it often convenient to assume a particular mathematical form for typically distributed measurements, and the form that is usually assumed is the normal distribution, so-called because it is very common in nature. The normal distribution corresponds to a bell-shaped curve which is symmetric about its peak, as illustrated in Figure 8.6. Repeated independent measurements of the same quantity (such as the breadth of an object, or its mass) approximate to a normal distribution. The more data is collected, the closer it will come to describing a normal distribution curve.

The peak of the normal distribution curve corresponds to the mean value of the distribution, as shown in Figure 8.7. This figure also illustrates how the standard deviation of a set of measurements is related to the spread. Although it is beyond the scope of this course to prove this, the area under a portion of a distribution curve within a certain range represents the number of measurements that lie within that range, as a proportion of the whole set. For a normal distribution, it turns out that 68% of the measurements lie within one standard deviation (i.e. within $\pm s_n$) of the mean value. Conversely, 32% of the measurements will lie outside this range. If you make a single additional measurement, it is therefore just over twice as probable that this one measurement will fall within one standard deviation of the mean than that it will fall outside this range. For a normal distribution, it also turns out that 95% of measurements are likely to fall within two standard deviations of the mean and 99.7% within three standard deviations of the mean.

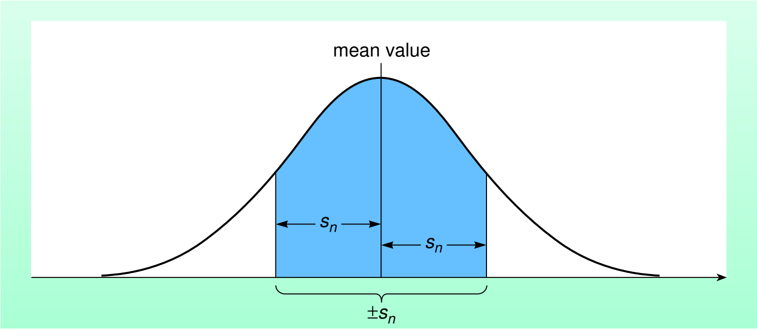


Figure 8.7: The shaded area under this normal distribution curve represents the measurements that lie within one standard deviation of the mean.

Remembering that precise measurements were defined in Section 8.2.1 as those for which the scatter was small, you will appreciate that the more precise a repeated set of the same number of measurements of a particular quantity, the more highly peaked the distribution curve and the smaller the standard deviation will be. A very broad distribution on the other hand, corresponds to measurements with considerable scatter and the standard deviation will be large. These trends are illustrated in Figure 8.8.

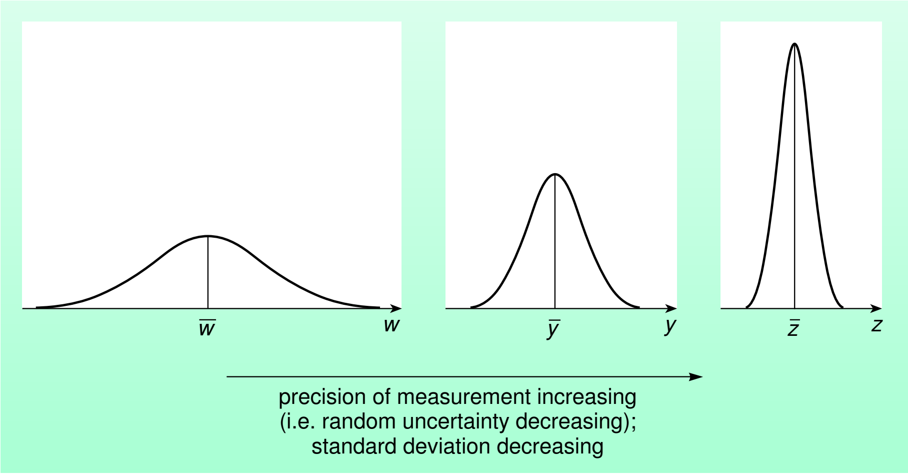


Figure 8.8: Normal distribution curves for three independent sets of measurements, with the same number of measurements in each set. The measurements of quantity w are subject to large random uncertainties, while those of quantity y are more precise and those of z more precise still.

8.2.6 Different types of 'average'

Figure 8.7 showed that if the data has a normal distribution the mean value corresponds to the peak of the distribution. Normal distributions of data are very common in science, but by no means universal. Figure 8.9 shows some other possible distributions, three of which are symmetric and one of which is skewed (i.e. not symmetric).

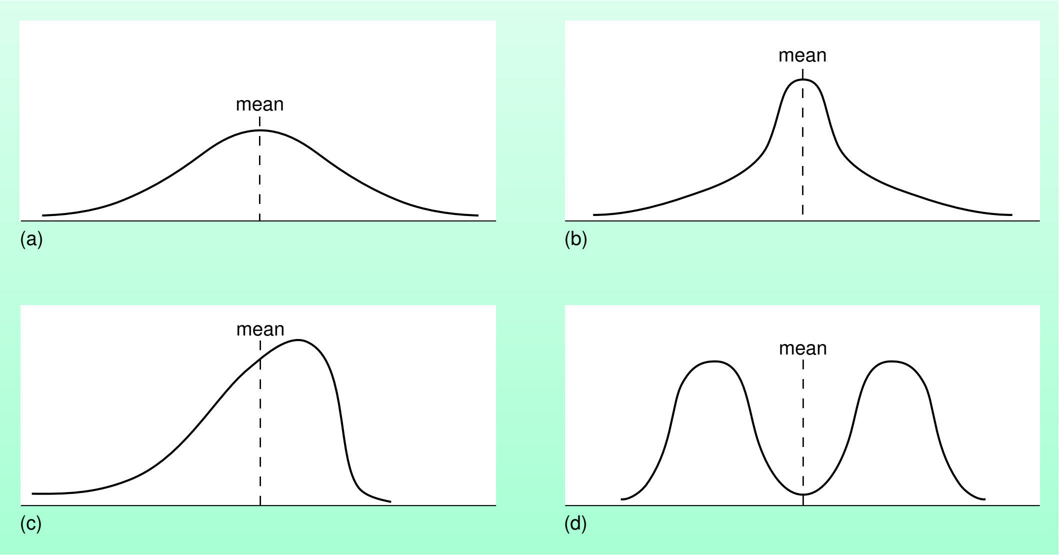


Figure 8.9: Types of distribution. (a) is the normal distribution: a symmetric bell shaped curve. (b) is also symmetric, but the shape of the curve does not approximate to the normal distribution. (c) is a skewed distribution. (d) is symmetric, but displays two equal maxima.

In many cases, especially if the distribution is skewed, the mean is not the best way of representing an average or typical value. Imagine for example a small company with a single owner who pays himself £900 000 a year and 10 employees who are each paid £10 000. The statement that the mean annual income of these 11 workers is more than £90 000 (i.e. £1000 000 divided by 11) — although true — is somewhat misleading! In such cases, two other quantities, the mode and the median, may represent the data more fairly.

The mode is the most frequently occurring value in the set of data. If the data is plotted on a histogram or a bar chart, the mode will be the value corresponding to the tallest bar.

Question

What is the mode of the earnings in the company described above?

Answer

The mode is £10 000. This is certainly more representative of the typical earnings than the mean would be!

Note that in some cases there may be more than one value for the mode; for example, that would be the case for the distribution shown in Figure 8.9d.

The median is the middle value in a series when the values are arranged in order of size. This means that half the measurements have values that are bigger than the median and half have values that are smaller than the median. If there are an odd number of measurements, the median is the middle measurement; if there are an even number of measurements it is the mean of the middle two values.

To see how this works, consider the following example. Ten plants of a particular species were chosen at random and the number of flowers on each plant were counted. The results were:

8; 7; 4; 8; 10; 7; 9; 7; 8; 7;

Question

What is the mode for this data?

Answer

The best way of answering this is to compile a table showing the number of plants with particular numbers of flowers:

number of flowers	4	7	8	9	10
number of plants	1	4	3	1	1

The mode is 7 flowers. There are more plants with 7 flowers than with any other number of flowers.

Question

What is the median for this data?

Answer

To answer this, we have to order the data. In increasing numbers of flowers, the results obtained were:

4; 7; 7; 7; 7; 8; 8; 8; 9; 10.

With a sample of 10 plants the median is the mean number of flowers on the 5$^{\text{th}}$ and 6$^{\text{th}}$ plants (counted in either ascending or descending order). In ascending order, the 5$^{\text{th}}$ plant has 7 flowers and the 6$^{\text{th}}$ has 8, so the median is $\frac{7+8}{2} = 7.5$.

Question 8.9

The heights of nine different specimens of the same type of plant were measured in centimetres, and the results in descending order were:

8.6; 8.3; 8.2; 7.9; 7.8; 7.8; 7.4; 7.3; 7.1

(a) What is the median of this data? Answer

(b) What is the mean of this data? Answer

Box 8.5 illustrates a case in which the mode and median give a more representative summary of the data than the mean.

Box 8.5 Seabird migration

In a study of Storm Petrels (small seabirds), several thousand birds were marked with identifying rings when they were at their nests on a Shetland island. After nesting, the birds dispersed. Twenty-eight of the birds were subsequently reported as having been recovered in other areas as shown in Table 8.5.

Taking all 28 observations into account, the mean distance from their nest site at which the birds have been recovered is 554.5 km. However, this is not a very useful way in which to summarize the data, because in fact 13 out of the 28 birds (i.e. nearly half) moved less than 100 km, and only two moved further than the mean distance. The median distance is 114 km and this is a more typical value.

Recovery place	Distance/km	Number of birds
Shetland (Lerwick)	49	8
Shetland (Foula)	77	5
Fair Isle	114	5
Orkney	157	2
Sule Skerry	248	3
Summer Isles	382	1
St Kilda	529	2
Cape Clear	1114	1
South Africa (Durban)	10568	1

Table 8.5: The recovery location of Storm Petrels ringed at their nests on one of the Shetland islands

This example shows how the mean can be highly dependent on a small number of measurements that are a long way from the mode. In this case, the single recovery from South Africa has an enormous influence on the mean. The median is 'resistant' to extreme values. Even if the bird recovered in South Africa had stopped in Morocco, or alternatively if it had gone to New Zealand, the median value would have remained 114 km.

8.2.7 Samples and populations

It is no accident that the examples used in Sections 8.2.3 and 8.2.4 to illustrate the statistics for repeated measurements of individual quantities were drawn from chemistry and physics. Experiments involving repeated measurements of some quantity are typical of the physical sciences. There are, however, many other types of scientific work in which a typical procedure is to collect data by measuring or counting the members of a sub-set of things which form part of a larger group, and Section 8.2.6 contained several examples. In this type of work, the sub-set of members that are measured or counted is called the sample and the larger group is called a population. Although often employed in the context of biology to describe a group of organisms that might breed with one another, the term ‘population’ is used much more widely in statistics to mean a collection of things or events. Examples of statistical populations could include all the sand grains on a beach, all the leaves on a tree, all the people in England with blood group AB, or all the visits made to the Science Museum in March.

It is generally the case that the members of any one population display some variability; for instance, not all the leaves on an oak tree will be exactly the same size. Furthermore, different populations often overlap with respect to whatever we might be measuring or counting. But despite this variability and overlap, what scientists often want to know is whether there seem to be systematic differences between the populations. Indeed, only if there do seem to be such differences do they accept that they really are dealing with more than one population. Failure to find evidence of systematic differences between the leaves of oak trees growing on sandy soil and those of oak trees growing on clay soils would suggest that the leaves (and trees) were members of the same population, or in other words that soil conditions have no overall effect on the leaves of oak trees. The statistical techniques used in looking for systematic differences between populations are the subject of Chapter 9, but in order to make use of these techniques it is necessary to be able to summarize the data that has been collected. You saw in Section 8.2.3 that for repeated measurements data sets could be summarized by quoting just two quantities: the mean and the standard deviation. This is also true for samples drawn from populations, but the mean and the standard deviation take on slightly different meanings in this context.

It is normally the case that data cannot be collected on all members of a population. It would indeed be impractical to attempt to measure every leaf on an oak tree! By the same token, it is usually impossible to know the *true* mean of some quantity for a whole population. This 'true mean' (also known as the 'population mean') is given the symbol μ (the Greek letter 'mew'), with the understanding that this quantity is generally not only unknown but unknowable. What we *can* easily calculate, however, is the mean of the quantity as measured for a sample drawn from the population. This is given the symbol $\overline{x}$ and calculated using Equation 8.3, exactly as we did in Section 8.2.3. Provided the sample is unbiased, $\overline{x}$ is the best estimate of μ that we can obtain.

As with the mean, the true standard deviation of a population can usually never be known with certainty. Again, the best estimate we can obtain must come from the distribution of values in a sample drawn from the population. However, this time it isn't appropriate to use the formula for the standard deviation of repeated measurements of one quantity which was:

$$s_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2} \tag{8.7}$$

Instead a slightly different formula is used, namely:

$$s_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2} \tag{8.8}$$

s_{n-1} is often called the 'sample standard deviation' because it is calculated from data taken for a sample of the population.

The value determined for s_{n-1} provides the best estimate of the standard deviation of the population. It will not have escaped your notice that the only difference between the two formulae is that in Equation 8.8 we are dividing by $(n-1)$, whereas in Equation 8.7 we were dividing by n. This means that s_{n-1} must always be larger than s_n (because we are dividing by a smaller number). This allows for the possibility that within the whole population there may be a few extremely high or low values of the measured quantity which will not necessarily be picked up in a sample drawn from that population.

s_{n-1} is also often called the 'estimated standard deviation of a population' because, provided the sample is chosen without bias, it is the best estimate that can be made of the true standard deviation of the population.

You should now check that you can use your calculator to determine the sample standard deviation s_{n-1} for a set of data. For this purpose, try taking the same set of numbers you used in Section 8.2.4 to check how to calculate s_n. These numbers were:

8, 6, 9, 12, 10.

The first four steps are the same as before, only Step 5 will be different.

Step 1

Put the calculator into statistical mode.

Step 2

Input all the data.

Step 3

If your calculator can tell you the number of items of data, check that it gives the answer '5' here.

Step 4

When you know you have the data correctly stored, display the mean; you should get the answer '9' here.

Step 5

Now find out how to display the sample standard deviation. The appropriate button will probably be marked σ_{n-1} or s_{n-1}. You should get the answer '2.2' here (to one decimal place). Don't use the σ_n or s_n button by mistake!

While this example is useful to familiarize yourself with the process, it doesn't represent a realistic scenario, not least because the hypothetical data set is so small. Because the aim is to estimate the mean and standard deviation for a whole population by carrying out measurements just on a sample, it is important to ensure that the sample is representative of the population as a whole and that usually requires it not only to be chosen without bias, but also to be reasonably large. In Question 8.10, the sample consists of 20 plants.

Question 8.10 Answer

Suppose that the number of flowers were counted on 20 orchid plants in a colony, and that the results were:

8; 8; 4; 8; 8; 7; 9; 7; 7; 5; 9; 10; 6; 9; 7; 4; 8; 5; 11; 5.

From this data, estimate to 3 significant figures the mean number, μ, of flowers per plant in the colony and the standard deviation of the population. You may if you wish construct a table similar to Table 8.3, but it will be much quicker simply to use your calculator.

8.3 Learning outcomes for Chapter 8

After completing your work on this chapter you should be able to:

8.1 demonstrate understanding of the terms emboldened in the text;

8.2 calculate the probability of a particular outcome from information about possible outcomes;

8.3 express a probability as a fraction, a decimal number or a percentage;

8.4 combine probabilities appropriately from information about possible outcomes;

8.5 interpret data in which the relative values of quantities are expressed as ratios;

8.6 calculate the mean, mode and median for a set of data;

8.7 calculate the standard deviation s_n for a set of repeated measurements of a particular quantity;

8.8 calculate the estimated standard deviation of a population, s_{n-1}, from a set of measurements made on a sample drawn from the population.

9 Statistical hypothesis testing

Samples and populations can be described in terms of their actual or estimated means and standard deviations, as discussed in Chapter 8. However, the ultimate aim of collecting data is usually not simply to describe, but also to answer scientific questions as objectively as possible. An extensive collection of statistical techniques has been developed over many years to provide answers to some such questions, bearing in mind that most data are intrinsically variable. In this chapter, you will be introduced to the general principles that underpin almost all of these techniques and then shown how to perform three particular statistical tests commonly used to answer different sorts of question. In order to illustrate these ideas and to provide some of the data with which you can practice, this chapter is partly based on a small ecological study. The study is described in Box 9.1.

Box 9.1 Green-winged Orchids and ridge-and-furrow topography

A conspicuous feature of parts of the English Midlands is ridge-and-furrow topography. Some of this is medieval and some is much later in age. It has been known for some time that Bulbous Buttercup (*Ranunculus bulbosus*) tends to occupy the drier ridges and Creeping Buttercup (*R. repens*) the wetter furrows. Also found in the same area is Green-winged Orchid (*Orchis morio*), a rare plant in England. A study was undertaken to find out whether the distribution and/or performance of Green-winged Orchid might also be influenced by ridge-and-furrow topography.

Various measurements were made on a sample of plants growing in a local nature reserve. Figure 9.1 illustrates some of the measurements taken. These included the horizontal and vertical distances of each plant from the nearest ridge crest, the height of the plant, the number of leaves and the number of flowers. Whether a plant was growing on the north-west or the south-east slope of the ridge was also recorded, since the two slopes might differ with respect to mean temperature, moisture availability, etc.

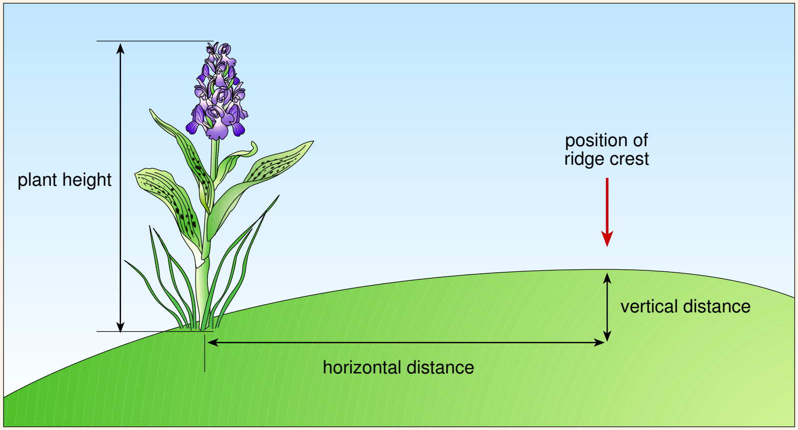


Figure 9.1: Measurement of horizontal and vertical distances of a plant from the nearest ridge crest and plant height.

9.1 The principles of hypothesis testing

Many of the questions that arise out of scientific investigations are driven by hypotheses, tentative explanations of observations that may be tested by experiment or by making further observations. Taking the study briefly described in Box 9.1 as an example, it might be proposed that Green-winged Orchid (like Bulbous Buttercup) occurs more frequently — and/or grows better — nearer the drier crests of ridges than the wetter furrows. Alternatively, it might be that Green-winged Orchid (like Creeping Buttercup) 'prefers' the wetter furrows to the drier ridges. Notice that these tentative ideas contain the unproven assumption that ridges are indeed drier than furrows. Statistical hypothesis testing provides a universally agreed set of procedures for answering questions such as 'Do Green-winged Orchids tend to occur nearer to ridge crests than expected by chance?', 'Does the amount of water in soil increase with distance from the nearest ridge crest?', 'Do the Green-winged Orchids growing nearer ridge crests tend to be taller or have more leaves and/or flowers than those growing further away?'.

There are two major branches of statistical hypothesis-testing: 'tests of association' (e.g. 'Are Green-winged Orchids found in association with ridge crests significantly more frequently than would be expected by chance?') and 'tests of difference' (e.g. 'Is there a significant difference between the mean height of plants growing on the north-west rather than the south-east slopes of ridges?').

Question

Would an investigation into whether there is a significant increase in the water content of soil with increasing distance from the nearest ridge crest be a test of association or a test of difference?

Answer

Since we would be looking to see if there is an *association* between soil water content and distance from ridge crest, this would be a test of association.

Having ascertained which statistical test is appropriate in any particular circumstance, most scientists look up the details of that test and then use it almost as if it were a 'black box' (a piece of equipment that users trust to perform a particular task reliably without understanding how it actually works). Sometimes, however, it is helpful to stand back from the details of any particular statistical test and to consider those features that are common to nearly all such tests. These common features can best be illustrated by considering in general terms a test of difference between two means.

Suppose that a scientist collected measurements from two samples of plants, one of which had been exposed to a particular experimental treatment and the other (the so-called control sample) which had not. Almost certainly, there would be some variation within each of these two sets of measurements and this would be reflected in their standard deviations. Moreover, even if the difference between the means of the experimental and control plants was relatively large, it would not be surprising if there was some overlap between the two sets of measurements, as shown in Figure 9.2.

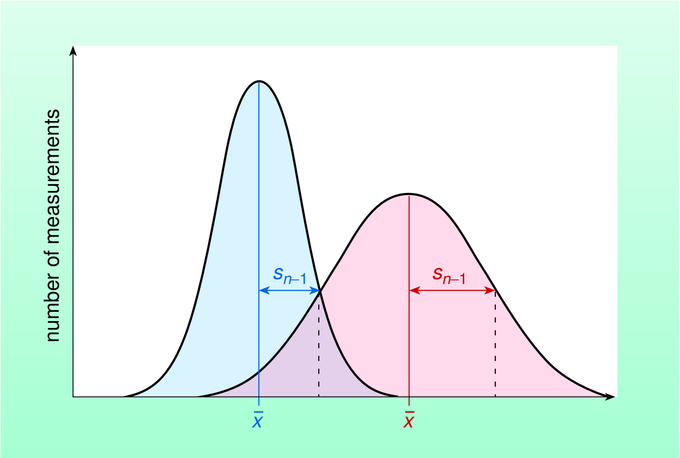


Figure 9.2: Diagram summarizing a possible result of an experiment in which a sample exposed to experimental treatment (shown with blue shading) was compared to a control (pink shading). Note the overlap between the two distributions. Each $\overline{x}$ is the mean of a sample (and estimates the mean of the population, μ) and each s_{n-1} is the sample standard deviation (estimated standard deviation of the population).
Note: The diagram shows the two distributions overlapping at exactly s_{n-1} from the mean. This would not normally be the case.

Now it might suit the scientist's favoured theory to convince others that the treatment *did* have a statistically significant effect on the measured character. On the other hand, it might be in the scientist's interests to show that the treatment *did not* have a statistically significant effect. Either way, the scientist is required by the procedures of statistical hypothesis testing to put forward a so-called null hypothesis in the first instance. As the name suggests, a null hypothesis is one of 'no difference'. In this case the null hypothesis would be that there is no difference between the population mean of the treated plants (μ_1) and the population mean of the control plants (μ_2). Expressing this statement mathematically, the null hypothesis would be that

$$\mu_1 = \mu_2$$

or, equivalently, that $\mu_1 - \mu_2 = 0$.

At the same time, the scientist has to put forward an alternative hypothesis that is the logical ‘mirror image’ of the null hypothesis. In this case the alternative hypothesis would be that there *is* a difference between the means of the treated and control plants. Expressing this statement mathematically, the alternative hypothesis would be that

$$\mu_1 \neq \mu_2$$

or $\mu_1 - \mu_2 \neq 0$.

Question

Is it possible for both the null and alternative hypotheses to be false?

Answer

No. If either is false, then the other must be true.

If the null hypothesis is true, then the alternative hypothesis *must* be false and vice versa.

Once statements of the null and alternative hypotheses have been made, a quantity called the test statistic is calculated. The test statistic is a number, on the basis of which a decision can be made to accept or reject the null hypothesis. The value of the test statistic depends on the characteristics of the samples being compared, and in most cases it is calculated using one or more equations. Things are often so arranged that the value of the test statistic comes out to be zero if the null hypothesis is true (for instance, by including the term $(\overline{x}_1 - \overline{x}_2)$ in the numerator of the equation, where $\overline{x}_1$ and $\overline{x}_2$, the means of the two samples, are the best available estimates of the unknowable values of μ_1 and μ_2). However, because of the vagaries of sampling, it would be extremely unlikely for the means of two samples drawn from even the same population to be identical (for instance, two samples of control plants are very unlikely to have exactly the same mean). So, the question is 'How large does the test statistic have to be before one can be reasonably confident that the samples were drawn from different populations (and therefore conclude, in this example, that the experimental treatment probably did have a significant effect)?' In fact, it is impossible to give a definitive answer to this question; it can be answered only in terms of probabilities.

Ideally, the precise probability that the calculated value of the test statistic could have arisen by chance if the null hypothesis were true would be determined. In a particular instance, this might turn out to be something like 1 in 63, i.e. $\frac{1}{63}$, which is 0.015 87 to four significant figures. In practice, the value of the test statistic is usually compared to lists of critical values calculated for a few pre-determined significance levels expressed in terms of probabilities. In this context, the probabilities are usually abbreviated to P and expressed in decimal notation, e.g. 0.1, 0.05 and 0.01. For any particular significance level, the critical value is the most extreme (usually largest) value that the test statistic could be expected to have if the null hypothesis were true. Of course, if the null hypothesis *is* true then any deviation from the test statistic's expected value (which, as noted above, is usually zero) must have arisen purely by chance. If the significance level corresponding to the value of the test statistic turns out to be quite low (usually because the test statistic is rather high), then it must be accepted that the null hypothesis is unlikely to be true. If the null hypothesis is false, then the alternative hypothesis *must* be true. Only at this stage can the scientist conclude:

- either that the treatment did have a significant effect (because the null hypothesis was probably false and therefore the alternative hypothesis probably true)
- or that the treatment did not have a significant effect (because the null hypothesis is likely to have been true).

It is extremely important to realize that the particular significance level at which a null hypothesis is rejected — and hence the alternative hypothesis is accepted — is a matter of convention. The usual convention in science is to reject a null hypothesis if the probability P is less than the 0.05 significance level, i.e. if $P < 0.05$. However, in employing this convention, it is also important to realize that you could either be rejecting a true null hypothesis or accepting a false one. Indeed, you are explicitly accepting that on average, if you were to carry out 100 statistical tests, you would reach the wrong conclusion for 5 of these tests (although you would not know which ones). If the work you are engaged in *really* matters, for example, medical research in which human lives might be at stake, then you would probably employ more exacting criteria (such as rejecting null hypotheses only if $P < 0.01$ or even $P < 0.001$). On the other hand, insisting on the use of such rigorous criteria for even routine scientific work would mean that many null hypotheses that really are false would have to be accepted, and this would undoubtedly hinder scientific progress.

The important features of statistical hypothesis testing are summarized below:

1. A null hypothesis (e.g. $\mu_1 = \mu_2$) and an alternative hypothesis (e.g. $\mu_1 \neq \mu_2$) are proposed.
2. The value of a test statistic is calculated.
3. If the probability of this value arising by chance if the null hypothesis were true is low — conventionally, less than 0.05 — then the null hypothesis is rejected and the alternative hypothesis accepted. (There is *always* the possibility of either rejecting a true null hypothesis or of accepting a false one.)

When null hypotheses are rejected, the results are described as being statistically significant or sometimes just as 'significant'. A consequence of this is often a feeling that 'non-significant' results are of less value than 'significant' ones. Indeed, there is probably a 'reporting bias' whereby significant differences are more likely to be published in scientific papers than non-significant ones. This undervaluing of non-significant differences is unfortunate because the whole point of the exercise is to try to find out what is happening in the real world. It may be just as important to know that an effect is *not* produced by one experimental treatment as to know that another treatment does produce the effect.

Question 9.1

Should the null hypothesis be accepted or rejected if the result of a statistical hypothesis test turned out to be:

(a) $P < 0.01$, Answer

(b) $P > 0.05$, Answer

(c) $P > 0.01$? Answer

9.2 Deciding which test to use; levels of measurement

The expression 'levels of measurement' refers to important distinctions between different sorts of data that might be collected during the course of a scientific investigation.

An example of data collected at the categorical level is the sex of animals. In most cases, an animal is unambiguously either 'male' or 'female'. Furthermore, there is no logical way in which the category 'male' can be ranked as 'higher' or 'better' than the category 'female' or vice versa. All that can be said is that these two categories are different. Of course, a data set may include more than two categories.

It *is* possible to rank ordinal level data in a sensible way. For instance, plants may be listed in order of their heights or grouped by the approximate number of leaves they possess without knowing the actual heights or the actual numbers of leaves. If the actual heights or numbers of leaves are known, then these data are at the interval level.

Data collected at the interval level can, if necessary, be analysed at the ordinal level. For instance, you might know that Plant A has 8 leaves and that Plant B has 5 leaves (interval level data). Nevertheless, you could choose to ignore some of this information and simply treat Plant A as having more leaves than Plant B (ordinal level data). Of course, if all you knew was that Plant A has more leaves than Plant B, then you could not convert this information into interval level data for analysis.

Categorical level data cannot usually be treated as if they were at interval or ordinal level (although you might argue that, for instance, red-flowered plants have more of a particular pigment than pink-flowered plants of the same species). However, by applying arbitrary criteria, interval or ordinal level data can sometimes be converted into categorical data for analysis. For instance, one of the seven pairs of contrasting characters used by Mendel in his pioneering research on the genetics of garden peas (see Box 8.2) was 'tall' versus 'short'. This categorical distinction made sense only because, in this particular case, there was no overlap between 'tall' and 'short' plants.

The reason for distinguishing between the different levels of measurement is that different statistical tests must be used to analyse categorical, ordinal and interval level data. Sometimes, when analysis of data at the interval level fails to reveal statistically significant differences, such differences may be shown up when the data are re-analysed at the ordinal level. However, because some information about the samples has effectively been 'thrown away' in the process, any statements eventually made about the populations from which the samples were drawn are necessarily less complete than they might have been.

Question 9.2

In each of the following cases, explain briefly whether the data should be treated as being at the categorical, the ordinal or the interval level.

(a) A count is made of the number of parasites on each member of a sample of sheep. Answer

(b) A sample of sheep are counted as either 'parasitized' (i.e. carrying one or more parasites) or 'unparasitized' (i.e. carrying no parasites). Answer

(c) A sample of sheep are counted as 'unparasitized' (i.e. carrying no parasites), 'lightly parasitized' (i.e. carrying 1–5 parasites), 'moderately parasitized' (i.e. carrying 6–10 parasites) or 'heavily parasitized (i.e. carrying more than 10 parasites). Answer

9.3 The χ^2-test

The χ^2-test (where χ is the Greek letter 'chi', said to rhyme with 'sky') is very commonly employed when scientists wish to test whether data on a single *categorical* variable match a particular theoretical pattern. Since 'presence' versus 'absence' is a categorical variable, the χ^2-test is often used to compare the numbers of individuals present in different areas with the numbers expected on the basis of an appropriate null hypothesis. This is more precisely called the χ^2 *goodness-of-fit* test. (There are other χ^2-tests, e.g. for possible associations between two categorical variables.)

In the Green-winged Orchid study, described in Box 9.1, horizontal distance from the nearest ridge crest (as shown in Figure 9.1) was recorded for 210 plants growing on several ridges. Because the ridge crest-to-furrow distance varied slightly between ridges, each of these distances was divided into five equal categories (category 1 being 0.00–19.9% of the distance from the crest, category 2 being 20.0–39.9% of the distance, category 3 being 40.0–59.9% of the distance, etc.) so that the data from different ridges could be pooled for analysis. This procedure enables us to treat interval level data (the horizontal distance of each plant from the nearest ridge crest) as categorical level data (the distance category into which each plant falls). If the 210 plants were distributed uniformly with respect to the ridge crest, then a fifth of them (i.e. 42) would be expected to occur within each distance category. A reasonable null hypothesis would be that, if it were possible to collect data on the entire population of Green-winged Orchids growing in fields with ridge-and-furrow topography, then there would be no difference between the number of plants observed in each distance category and the number that would be expected on the assumption that the plants were distributed uniformly. The alternative hypothesis would be that the number of plants observed in each distance category was *not* equal to the number of plants expected. Accepting this alternative hypothesis implies accepting that the plants were distributed non-uniformly.

In fact, of the sample of 210 plants, 105 occurred in the first distance category, 74 in the second, 28 in the third, 3 in the fourth and none in the fifth. It certainly *appears* that the plants were not uniformly distributed. The χ^2-test allows a definitive statement to be made on the probability that the population of plants from which the sample was drawn *could* have been distributed uniformly despite the apparently non-uniform distribution observed in the sample. Only if this probability is sufficiently low (conventionally if $P < 0.05$) can the null hypothesis be rejected and the alternative hypothesis (with its implication that the plants were distributed non-uniformly) accepted.

The first stage in performing a χ^2-test is usually to draw up a table to compare observed and expected numbers in different categories. The table for the sample of 210 orchid plants is given in Table 9.1, and compares the number of individuals, O_i, that were observed in each distance category, with the number E_i expected on the basis of the null hypothesis. As a check, the total number in the O_i column should equal the total number in the E_i column. The trickiest part of most χ^2-tests is deciding the 'expected' numbers. In this case, if the null hypothesis were true, a fifth of the plants (i.e. 42) would be expected to fall into each distance category.

Distance category	Observed number (O_i)	Expected number (E_i)
1 (nearest to ridge)	105	42
2	74	42
3	28	42
4	3	42
5 (furthest from ridge)	0	42
total	210	210

Table 9.1: Table comparing the observed distribution of a sample of Green-winged Orchids across 5 categories of distance from the nearest ridge crest with the distribution expected if the plants were distributed uniformly

The test statistic is χ^2 and this is found in the following way.

1. For each distance category, the 'expected' number is subtracted from the 'observed' number.

   This gives $(O_i - E_i)$.

2. Each result from step 1 is squared.

   This gives $(O_i - E_i)^2$.

3. Each result from step 2 is divided by the appropriate 'expected' number.

   This gives $\dfrac{(O_i - E_i)^2}{E_i}$.

4. The results from step 3 are totalled.

   This gives $\displaystyle\sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$ which is the test statistic χ^2.

In summary,

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \tag{9.1}$$

The easiest way to calculate χ^2 is to extend Table 9.1 to include columns for each of these steps. This has been done in Table 9.2, and χ^2 is the total of the values in the right-hand column. Notice that, as a further check, the total of the $(O_i - E_i)$ column must be zero, since the total number of individuals observed is equal to the total number of individuals expected.

Distance category	O_i	E_i	$(O_i - E_i)$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
1	105	42	63	3969	94.500
2	74	42	32	1024	24.381
3	28	42	−14	196	4.667
4	3	42	−39	1521	36.214
5	0	42	−42	1764	42.000
total	210	210	0		201.762

Table 9.2: Extension of Table 9.1 to calculate χ^2.

As an example of the way in which each value is calculated, consider the first distance category. $O_i = 105$ and $E_i = 42$ so

$$\frac{(O_i - E_i)^2}{E_i} = \frac{(105 - 42)^2}{42} = \frac{63^2}{42} = \frac{3969}{42} = 94.500$$

This value, and all the other values in the right-hand column of Table 9.2 have been calculated to three places of decimals. This is normal practice when finding χ^2 because this is how values are generally stated in tables of critical values for χ^2.

The next stage is to compare the value of the test statistic χ^2 (which, in this case, is 201.762) with the critical values listed in Table 9.3. The sizes of the critical values in such a table depend on both the significance level ($P = 0.1$, $P = 0.05$ and $P = 0.01$, given across the top of the table) and the number of degrees of freedom (given down the left-hand side of the table).

The number of degrees of freedom can be found by counting the number of 'cells' in the table that contain observed counts (i.e. ignoring expected counts, totals etc.).

> For the χ^2-test, the number of degrees of freedom is given by
>
> number of cells – 1

In this case, Table 9.1 has five cells, so

$$\begin{aligned}\text{number of degrees of freedom} &= 5 - 1\\ &= 4\end{aligned}$$

Box 9.2 gives a brief explanation of why it is reasonable for the number of degrees of freedom to be four in this case.

Box 9.2 Degrees of freedom

Why should the number of degrees of freedom be four for the data given in Table 9.1? The total numbers of both 'observed' and 'expected' plants became fixed (in this case, at 210) the moment data collection ceased. The number of expected plants in each of the distance categories (42) is fixed by a combination of the null hypothesis being tested (i.e. that equal numbers of plants would be expected in each distance category) and the sample size (i.e. 210). In contrast, the number of plants that could have been observed in each of any four of the distance categories is completely free to vary, although the number of plants that could have been observed in the final category is *not* free to vary in this way — it must be such that the total of the numbers in the observed column equals the sample size (i.e. 210). In this case, there are therefore four degrees of freedom.

Similar arguments to the above underpin the concept of degrees of freedom in other statistical tests.

The parts of Table 9.3 that are relevant to our example are reproduced in Table 9.4. Reading across the row for 4 degrees of freedom, it can be seen that the χ^2 value of 201.762 is greater than 7.779 (corresponding to a significance level of 0.1), greater than 9.488 (corresponding to a significance level of 0.05) and greater than 13.277 (corresponding to a significance level of 0.01). In fact, the significance level is considerably less than 0.01 (because 201.762 is *much* larger than 13.277). Thus, the probability that the plants in the population from which the sample was drawn were distributed uniformly is *much* less than 0.01 (i.e. $P \ll 0.01$). There can be little doubt that the plants were not distributed uniformly with respect to distance from the ridge crest. The null hypothesis can therefore be rejected — and the alternative hypothesis accepted — with a great deal of confidence. In reporting such a result, it is often stated that the null hypothesis is rejected at the $P = 0.01$ significance level or (probably more commonly) at the 1% significance level.

Degrees of freedom	$P = 0.1$	$P = 0.05$	$P = 0.01$
1	2.706	3.841	6.635
2	4.605	5.991	9.210
3	6.251	7.815	11.341
4	7.779	9.488	13.277
5	9.236	11.070	15.086

Table 9.4: Part of Table 9.3.

Although statistics shows that the plants were distributed non-uniformly, it does not reveal the *nature* of the non-uniform distribution. The data should now be re-inspected to confirm that the plants did indeed occur *closer* to the ridge crests than expected by chance — rather than nearer to the furrows or clustered halfway between the ridges and furrows. The conclusion that can be drawn from this investigation is that Green-winged Orchids tend to occur significantly closer to ridge crests than to furrows.

Precautions for the χ^2-test

You will not be expected, in this course, to decide which statistical test to use in any given situation. However, in general, before performing a χ^2-test you should check that:

- the data are at the categorical level;
- the 'observed' numbers are actual counts (*not* proportions or percentages);
- none of the 'expected' numbers is less than 5 (a design feature of the test).

Worked example 9.1 shows the use of a χ^2-test in investigating whether or not an observed distribution of organisms is consistent with a particular theoretical *ratio.* Questions of this type are quite common, and the first step is always to work out the *number* of organisms expected in each category if the null hypothesis — that the theoretical ratio holds — is true. The worked example also illustrates that, while observed numbers of organisms must always be whole numbers, the numbers expected on the basis of theory or prediction often come out to be fractions.

Worked example 9.1

A biologist makes the prediction that flies of type A, type B and type C will occur in the ratio 0.16 : 0.48 : 0.36 in a wild population, *if* this population is in so-called Hardy–Weinberg equilibrium. A representative sample drawn from a population was found to contain 28 type A flies, 134 type B flies and 78 type C flies. Was this population in Hardy–Weinberg equilibrium?

Answer

The total number of flies in the sample was $28 + 134 + 78 = 240$. If the ratio in the sample was 0.16 type A flies : 0.48 type B flies : 0.36 type C flies, then there would be

$$0.16 \times 240 = 38.4 \text{ type A flies}$$
$$0.48 \times 240 = 115.2 \text{ type B flies}$$
$$0.36 \times 240 = 86.4 \text{ type C flies}$$

These are therefore the 'expected' numbers.

A table, extended to give values for $(O_i - E_i)$, $(O_i - E_i)^2$ and $\dfrac{(O_i - E_i)^2}{E_i}$ is given in Table 9.5.

Fly type	O_i	E_i	$(O_i - E_i)$	$(O_i - E_i)^2$	$\dfrac{(O_i - E_i)^2}{E_i}$
A	28	38.4	–10.4	108.16	2.817
B	134	115.2	18.8	353.44	3.068
C	78	86.4	–8.4	70.56	0.817
total	240	240	0		6.702

Table 9.5: An extended table for Worked example 9.1

The number of degrees of freedom is given by

$$\begin{aligned}&\begin{pmatrix}\text{number of cells containing}\\ \text{observed numbers}\end{pmatrix} - 1\\ &\quad = 3 - 1\\ &\quad = 2\end{aligned}$$

Reading across the row for 2 degrees of freedom in Table 9.3, it can be seen that the χ^2 value of 6.702 corresponds to a significance level of less than 0.05 but more than 0.01 (i.e. $0.05 > P > 0.01$).

The probability that the ratio of different types of fly in the entire population from which the sample of 240 was drawn is 0.16 type A : 0.48 type B : 0.36 type C is less than 0.05. This means that the null hypothesis (that the population is in Hardy–Weinberg equilibrium) must be rejected at the 5% significance level. On the basis of this investigation, it must be concluded that the population is *not* in Hardy–Weinberg equilibrium.

Question 9.3 Answer

The prediction is made on the basis of theory that, if a particular genetic cross were to be performed, the ratio of plants in the next generation should be 1 red-flowered : 2 pink-flowered : 1 white-flowered.

The next generation of a sample comprised 185 red-flowered plants, 305 pink-flowered plants and 146 white-flowered plants. Is this data compatible with the 1 : 2 : 1 ratio predicted?

9.4 The Spearman rank correlation coefficient

In the study described in Box 9.1, soil samples were taken right across a ridge at different horizontal distances from the crest, in order to test whether the water content of the soil varies significantly from ridge crests to furrows. Both the original mass of each sample and its mass after drying in an oven were measured using a scientific balance. The water content of each sample was then expressed as a percentage of its dry mass. For example, since the original mass of one sample was 22.85 g and its dry mass was 11.32 g, its water content was

$$\frac{(22.85\text{ g} - 11.32\text{ g})}{11.32\text{ g}} \times 100\% = 102\%$$

(This percentage is greater than 100% because there was slightly more water than soil in the sample.)

In fact, several soil samples (known as ‘replicate’ samples) were taken at each horizontal distance, and their mean water content was calculated and used for the rest of the investigation. Figure 9.3 shows how the mean water content of the soil samples taken on the north-west slope of the ridge varied with horizontal distance from the nearest ridge crest.

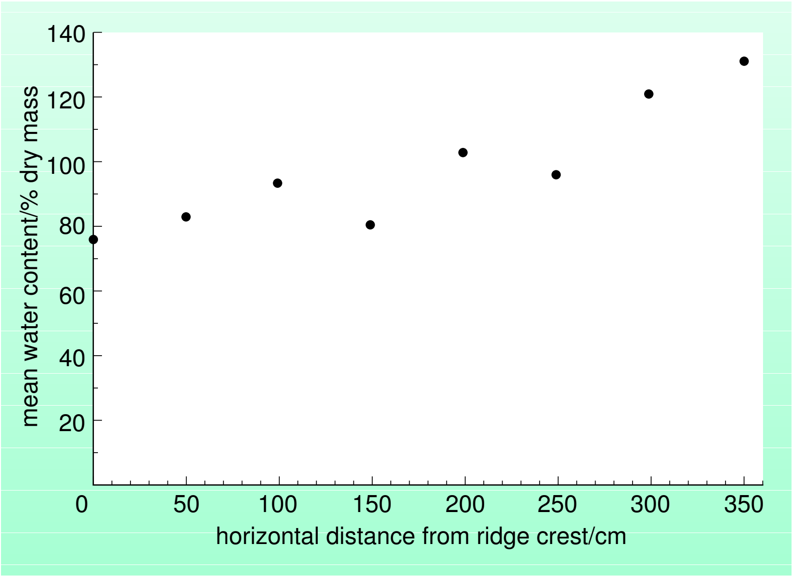


Figure 9.3: Mean water content (as a percentage of dry mass) of soil samples plotted against horizontal distance from ridge crest.

There certainly seems to be a trend of water content increasing with increasing horizontal distance from ridge crest. But is this trend, or apparent correlation between these two variables, statistically significant? The strength of a possible correlation between two variables is summarized in the value of a correlation coefficient (r). The value of r can range from +1 (i.e. a perfect positive correlation, in which the two variables increase or decrease precisely in step with one another; Figure 9.4a) to −1 (i.e. a perfect negative correlation, in which one variable increases as the other decreases and vice versa; Figure 9.4b). Where there is no correlation between two variables, the value of r is zero (Figure 9.4c). Figure 9.3 suggests that, for mean soil water content and horizontal distance from nearest ridge crest, r lies somewhere between 0 and +1. However, we need to determine the actual value of r and hence determine the probability that — for the population of all possible soil water contents — the null hypothesis (that there is no correlation between water content and horizontal distance from ridge crest) is true.

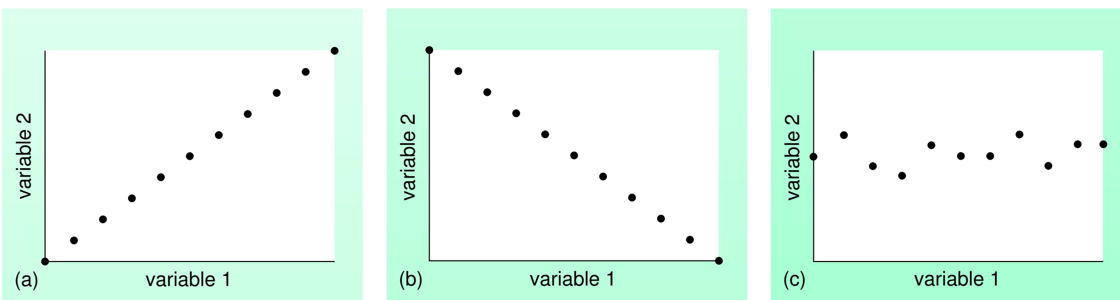


Figure 9.4: (a) A perfect positive correlation between two variables (i.e. $r = +1$). (b) A perfect negative correlation (i.e. $r = -1$). (c) No correlation (i.e. $r = 0$). A graph with points scattered over it in a random way also represents zero correlation.

Several different sorts of correlation coefficient have been devised. In this case it is appropriate to calculate the Spearman rank correlation coefficient (r_s). This, as the term 'rank' suggests, is based on ordinal level data. The null hypothesis is that there is no correlation between soil water content and horizontal distance from ridge crest (i.e. $r_s = 0$) and the alternative hypothesis that the two variables are correlated (i.e. $r_s \neq 0$).

The measurements of mean soil water content for the north-west slope of the ridge are summarized in Table 9.6.

Horizontal distance/cm	Mean water content/% dry mass
0	76
50	83
100	93
150	80
200	102
250	95
300	120
350	130

Table 9.6: Mean soil water content (as percentage of dry mass) for samples taken at various horizontal distances from the nearest ridge crest on the north-west slope of a ridge

Before the test statistic can be calculated, the following steps should be completed:

1. Work out the rank (order) of each of the 8 horizontal distances, $(R_A)_i$ (which will range between 1 and 8).

2. Work out the rank of each matching value for mean water content, $(R_B)_i$ (which will also range between 1 and 8).

3. Calculate each difference, $D_i = (R_A)_i - (R_B)_i$.

4. Square each difference, to give D_i^2.

5. Total all the values for D_i^2 from Step 4 to give $\sum_{i=1}^{n} D_i^2$.

As an example of Steps 1 to 4, consider the horizontal distance 150 cm, which has $(R_A)_i = 4$ and $(R_B)_i = 2$ (since its distance is fourth from the crest while its water content is second lowest).

Therefore, $D_i = (R_A)_i - (R_B)_i = 4 - 2 = 2$.

So $D_i^2 = 2^2 = 4$.

The other values for D_i^2 are shown in Table 9.7, and the total of the numbers in the right-hand column of this table gives $\sum_{i=1}^{n} D_i^2$. Notice that $\sum_{i=1}^{n} D_i$ (the sum of the differences of the ranks) should always be zero, which provides a check that the ranks have been worked out correctly.

Horizontal distance/cm	Rank $(R_A)_i$	Mean water content/%	Rank $(R_B)_i$	$D_i = (R_A)_i - (R_B)_i$	D_i^2
0	1	76	1	0	0
50	2	83	3	–1	1
100	3	93	4	–1	1
150	4	80	2	2	4
200	5	102	6	–1	1
250	6	95	5	1	1
300	7	120	7	0	0
350	8	130	8	0	0
				$\sum_{i=1}^{n} D_i = 0$	$\sum_{i=1}^{n} D_i^2 = 8$

Table 9.7: Extension of Table 9.6 to include ranks

In the case of the data in Table 9.7, it was possible to assign a unique rank to each value for horizontal distance and mean water content, but sometimes quantities 'tie' (i.e. have the same rank). Worked example 9.2, at the end of this section, illustrates what to do when this is the case.

The test statistic, the Spearman rank correlation coefficient, is r_s and this is calculated using Equation 9.2:

$$r_s = 1 - \frac{6 \sum_{i=1}^{n} D_i^2}{n(n^2 - 1)} \tag{9.2}$$

where $\sum_{i=1}^{n} D_i^2$ is the sum of the squares of the differences of the ranks and n is the number of *pairs* of measurements.

Substituting $\sum_{i=1}^{n} D_i^2 = 8$ (from Table 9.7) and $n = 8$ into Equation 9.2 gives

$$r_s = 1 - \frac{6 \times 8}{8 \times (8^2 - 1)} = 0.905$$

The final stage is to compare the value of the test statistic r_s (0.905 in this case) with the critical values listed in Table 9.8. The critical values are again given to three places of decimals and the size of the critical values depends on both the significance level ($P = 0.1$, $P = 0.05$ and $P = 0.01$, given across the top of the table) and the number of *pairs* of measurements (given down the left-hand side of the table). In this case the number of pairs of measurements is 8, and looking across the appropriate row it can be seen that the calculated r_s value of 0.905 is greater than

0.881, corresponding to a significance level of 0.01. Thus the probability, P, that there is no correlation between water content and horizontal distance from the ridge crest is less than 0.01; the null hypothesis must be rejected at the 1% significance level, and the alternative hypothesis accepted. There is a statistically significant (positive) correlation between mean soil water content and horizontal distance from ridge crest.

It is extremely important to appreciate that even a statistically significant correlation between two variables does *not* prove that changes in one variable *cause* changes in the other variable.

Correlation does not imply causality.

A time-honoured, but probably apocryphal, example often cited to illustrate this point is the statistically significant positive correlation reported for the late nineteenth century between the number of clergymen in England and the consumption of alcoholic spirits. Both the increased number of clergymen and the increased consumption of spirits can presumably be attributed to population growth (which is therefore regarded as a 'confounding variable') rather than the increase in the number of clergymen being the *cause* of the increased consumption of spirits or vice versa!

Precautions for the Spearman rank correlation test

Before calculating a Spearman rank correlation (r_s) it is necessary to check that:

- the data was collected at, or can be converted into, ordinal level (i.e. ranks);
- there are 7 to 30 pairs of measurements (though the test can be performed with more than 30 pairs if you have access to a more extensive table of critical values);
- these measurements are reasonably scattered.

Worked example 9.2 illustrates how to rank data when two or more measurements are identical. They must be given the same mean rank, and then account must be taken of all the identical measurements before the rank of the next, non-identical, value is decided. So, if two measurements tie for first place, they are each given a rank of $\left(\frac{1+2}{2}\right) = 1.5$, and the next available rank is 3.

Worked example 9.2

The number of Stonefly nymphs counted in standard samples taken at 13 stations along a stream, together with the water speed measured at these stations, is presented in Table 9.9. Calculate the Spearman rank correlation coefficient (r_s) for this data and use this to determine whether there is a statistically significant correlation between water speed and the number of Stonefly nymphs present.

Answer

Table 9.10 is an extension of Table 9.9, to include values for $(R_A)_i$, $(R_B)_i$, D_i, D_i^2 and $\sum_{i=1}^{n} D_i^2$ for the data in this worked example.

Note, for example, that the water speed was measured to be 0.2 m s^{-1} at two sampling stations, so these stations 'tie' for second place in the ranking of water speed (after the station with a water speed of 0.1 m s^{-1}). Each is given a rank of $\left(\frac{2+3}{2}\right) = 2.5$, and the next available rank (for the station with a water speed of 0.4 m s^{-1}) is 4.

Water speed/m s^{-1}	Number of nymphs
0.8	35
1.1	28
0.5	11
0.7	12
0.2	7
0.4	5
0.5	6
1.3	21
0.9	23
1.7	43
0.2	10
0.1	6
0.7	19

Table 9.9: Number of Stonefly nymphs in relation to the speed of water flow at 13 sampling stations in a stream

Substituting $\sum_{i=1}^{n} D_i^2 = 47.5$ (from Table 9.10) and $n = 13$ into Equation 9.2:

$$r_s = 1 - \frac{6 \times 47.5}{13 \times (13^2 - 1)} = 0.870$$

Reading across the row for 12 pairs of measurements (in the absence of a row for 13 pairs) in Table 9.8, it can be seen that $P < 0.01$. The null hypothesis must therefore be rejected at the 1% significance level and the alternative hypothesis accepted. There is a statistically significant positive correlation between water speed and number of Stonefly nymphs.

Question 9.4 Answer

Returning to the study described in Box 9.1, Figure 9.5 shows how the mean water content of the soil samples taken from the north-west slope of the ridge varies with *vertical* distance from ridge crest. Use the data given in Table 9.11 to determine whether there is a statistically significant correlation between soil water content and vertical distance from ridge crest.

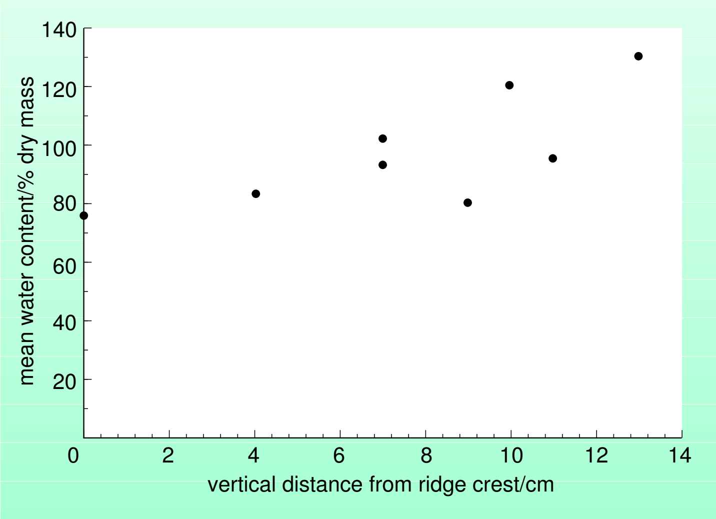


Figure 9.5: Mean water content (as a percentage of dry mass) of soil samples plotted against vertical distance from ridge crest.

Vertical distance/cm	Mean water content/ % dry mass
0	76
4	83
7	93
9	80
7	102
11	95
10	120
13	130

Table 9.11: Vertical distances from the nearest ridge crest and mean soil water content (as a percentage of dry mass) for samples taken at various horizontal distances from the nearest ridge crest on the north-west slope of a ridge

9.5 The *t*-test for unmatched samples

Several *t*-tests are widely used to test whether the means of two samples are sufficiently different to conclude that the samples were probably drawn from different populations. Such a conclusion might allow an experimenter to conclude further that, for example, an experimental treatment did produce a statistically significant effect compared to the experimental control (Section 9.1). *t*-tests are often referred to as 'Student's *t*-tests'. This is not because people such as yourself use them a lot — although this is true! 'Student' was the pseudonym used by W. S. Gossett when he published the first version of the test in 1907. His employer, a well-known brewing company based in Dublin, would not allow him to publish under his own name.

Question

State the null and alternative hypotheses that would be appropriate for a *t*-test.

Answer

Since a *t*-test would be concerned with the difference between the *means* of two *populations* (1 and 2), the appropriate null hypothesis would be either $\mu_1 = \mu_2$ or its equivalent $\mu_1 - \mu_2 = 0$ and the appropriate alternative hypothesis either $\mu_1 \neq \mu_2$ or $\mu_1 - \mu_2 \neq 0$ (see Section 9.1).

As indicated by the section heading, the *t*-test introduced here is specifically for *unmatched* samples. It is therefore necessary to discuss what is meant when samples are said to be either ‘matched’ or ‘unmatched’.

The soil samples (and hence their mean water content) discussed in Section 9.4 were uniquely matched to particular horizontal distances from the nearest ridge crest. If data was collected from *individual* patients before and after they were given either an experimental medicine or a placebo (i.e. a ‘dummy’ medicine), this data would also be matched. Another example of matched samples would be the test scores achieved by individual employees before and after a training event.

A typical situation in which a t-test for unmatched samples would be used is if the heights of two samples of Green-winged Orchids were measured, one sample growing on the north-west slope of a ridge and the other sample growing on the south-east slope (Table 9.12). Since there is no logical connection between any one plant growing on the north-west slope and any one plant growing on the south-east slope, these two samples are unmatched.

	North-west slope (i.e. Sample 1)	South-east slope (i.e. Sample 2)
$\overline{x}$/cm	18.6	21.1
s_{n-1}/cm	5.5	3.9
n	14	16

Table 9.12: Mean plant height ($\overline{x}$), estimated population standard deviation (sample standard deviation) of plant height (s_{n-1}) and sample size (n) for a sample of plants growing on the north-west slope of a ridge (Sample 1) and another sample growing on the south-east slope (Sample 2).

Question 9.5

In each of the following cases, explain whether the samples are matched or unmatched.

(a) A comparison is made between the heights of a sample of Green-winged Orchids growing in one nature reserve and those of a sample growing in another nature reserve. Answer

(b) The numbers of nymphs of two species of Stonefly are counted in each of 10 samples taken at different positions along a stream. Answer

In order to calculate the test statistic in this particular *t*-test it is necessary to solve three equations one after another. The test statistic itself is t and this is calculated using Equation 9.3, in which $\overline{x}_1$ and $\overline{x}_2$ are the means of the two samples, 1 and 2, that may, or may not, have been drawn from different populations.

$$t = \frac{\overline{x}_1 - \overline{x}_2}{SE_D} \tag{9.3}$$

Notice that, if $\overline{x}_1 = \overline{x}_2$ (which would mean that $\overline{x}_1 - \overline{x}_2 = 0$), then $t = 0$. So, if the null hypothesis were true, then it would be expected that $t = 0$. The term SE_D represents the 'standard error of the differences in the sample means'. SE_D is calculated using Equation 9.4, in which n_1 and n_2 are the two sample sizes.

$$SE_D = \sqrt{\frac{(S_c)^2}{n_1} + \frac{(S_c)^2}{n_2}} \tag{9.4}$$

The term $(S_c)^2$ (which appears twice in Equation 9.4) represents the 'common population variance'. $(S_c)^2$ is calculated using Equation 9.5, in which s_1 and s_2 are the two estimated population standard deviations (also known as sample standard deviations, as discussed in Section 8.2.7). In fact, each of s_1 and s_2 should really be written as s_{n-1}, but if this were done the subscripts would be getting out of hand!

$$(S_c)^2 = \frac{(n_1 - 1)(s_1)^2 + (n_2 - 1)(s_2)^2}{(n_1 - 1) + (n_2 - 1)} \tag{9.5}$$

Inspection of Equation 9.3 shows that, other things being equal, the greater the difference between $\overline{x}_1$ and $\overline{x}_2$, the larger the value of t. In addition, if the sample means are well separated, it seems reasonable to expect that there is likely to be a statistically significant difference between the true means of the populations from which the samples were drawn. Similar arguments can be used to link small values of s_1 and s_2 and large values of n_1 and n_2 to both high values of t and an increased likelihood of a statistically significant difference between the means from which the samples were drawn. In general, high values of t are associated with greater statistical significance.

Returning to the data summarized in Table 9.12, notice that the mean height of the sample 2 (21.1 cm) is greater than that of sample 1 (18.6 cm). What needs to be established is whether or not the difference observed (2.5 cm) is statistically significant.

Substituting the relevant values into Equation 9.5:

$$S_c^2 = \frac{(14-1)(5.5\text{ cm})^2 + (16-1)(3.9\text{ cm})^2}{(14-1)+(16-1)}$$
$$= \frac{(13 \times 30.25\text{ cm}^2) + (15 \times 15.21\text{ cm}^2)}{13+15}$$
$$= 22.193\text{ cm}^2$$

Substituting the relevant values into Equation 9.4:

$$SE_D = \sqrt{\frac{22.193\text{ cm}^2}{14} + \frac{22.193\text{ cm}^2}{16}}$$
$$= 1.724\text{ cm}$$

Substituting the relevant values into Equation 9.3:

$$t = \frac{18.6\text{ cm} - 21.1\text{ cm}}{1.724\text{ cm}}$$
$$= -1.450$$

What does a value of $t = -1.450$ mean? Did the populations of plants growing on the north-west and south-east slopes of this ridge really differ in mean height or could the observed difference in mean height between the two samples (i.e. 2.5 cm) have arisen by chance?

The fact that the test statistic t turns out to have a negative value can be ignored. If it happened that the mean height of the sample of plants growing on the north-west slope of the ridge had been deducted from that of the sample growing on the south-east slope, rather than the other way around, then t would have been +1.450. Only the absolute value of t (i.e. the number without its sign, in this case 1.450) is of any consequence.

The critical values of t are given in Table 9.13.

For the t-test for unmatched samples, the number of degrees of freedom is given by

$$(n_1 - 1) + (n_2 - 1)$$

Since in this case n_1 is 14 and n_2 is 16, the number of degrees of freedom is

$$(14 - 1) + (16 - 1) = 13 + 15 = 28$$

Reading across the row corresponding to 28 degrees of freedom to find the highest critical value exceeded by the value of the test statistic (i.e. 1.450), it can be seen that all that can said is that $P > 0.1$. Since P is *not* less than 0.05, the null hypothesis (that $\mu_1 = \mu_2$) must be accepted and the alternative hypothesis (that $\mu_1 \neq \mu_2$) rejected. There is therefore no evidence that the samples were taken from different populations of plants. The plants growing on the north-west and south-east slopes of this ridge do not differ statistically significantly from one another in mean height.

Precautions for the t-test for unmatched samples

Before performing a t-test for unmatched samples it is necessary to check that:

- the samples are unmatched (if the samples are matched, then a different version of the t-test must be used);
- population means are to be compared (different statistical tests must be used if population modes or medians are to be compared);
- the data are at the interval level (again, different statistical tests must be used if the data are at either ordinal or categorical level);
- there are fewer than about 25 items of data in each sample (if the samples are larger than this, then a different — more straightforward! — statistical test known as a z-test should be used);
- the assumption can be made that the population(s) from which the samples were drawn have normal distributions and approximately equal standard deviations.

Question 9.6 Answer

Descriptive statistics on the number of flowers per plant for samples of plants growing on the north-west and south-east slopes of another ridge are given in Table 9.14. Is there a statistically significant difference between the slopes in the mean number of flowers per plant?

	North-west slope (i.e. Sample 1)	South-east slope (i.e. Sample 2)
$\overline{x}$	7.7	7.2
s_{n-1}	2.7	2.1
n	18	15

Table 9.14: Mean number of flowers per plant ($\overline{x}$), estimated population standard deviation of number of flowers per plant (s_{n-1}) and sample size (n) for a sample of plants growing on the north-west slope of a ridge (Sample 1) and another sample growing on the south-east slope (Sample 2).

9.6 Other statistical tests

You have been introduced to three particular statistical hypothesis tests in Sections 9.3–9.5. Over the years, many tests have been devised to perform a wide range of statistical tasks in the context of science. Some of these tests (for example the *t*-test for *matched* samples and the χ^2-test for association) are similar to those covered here, but most are designed to answer different sorts of scientific questions or to be used in rather different circumstances.

Many excellent books have been written to help you select which particular statistical test is most appropriate for the task at hand and then guide you through performing that test. Sections 9.1 and 9.2 of this chapter should enable you get to grips quickly and relatively painlessly with unfamiliar statistical tests when the time comes for you to branch out.

9.7 Learning outcomes for Chapter 9

After completing your work on this chapter you should be able to:

9.1 demonstrate understanding of the terms emboldened in the text;

9.2 propose null and alternative hypotheses in familiar circumstances;

9.3 perform a χ^2-test and interpret the results;

9.4 calculate a Spearman rank correlation coefficient (r_S) and then test its statistical significance;

9.5 perform a *t*-test for unmatched samples and interpret the results.

Differentiation

10

In science, and in everyday life, we often want to know how one quantity varies with respect to another. We may be interested in the actual value of one quantity for a particular value of the other quantity, but it is often the *rate* at which one quantity varies with respect to another that is of more importance. Consider, for example, a small plant. The height of the plant as we look at it is of some interest, but we also want to know whether the plant is growing, and if so, how fast. Also, is the plant growing at an ever increasing rate or is its rate of growth slowing down? If the growth is slowing down the plant may fit in the space we've made for it on the windowsill; if the rate of growth is increasing we may need to think again!

Chapter 5 introduced the concept of the gradient of a graph as a way of finding rate of change, whether that be positive (as in Figure 5.9), negative (Figure 5.15) or zero (Figure 5.16 for Object B). However, Chapter 5 considered the gradient of straight-line graphs only; we need to extend the concept to enable us to find the gradient of curves.

Section 10.1 discusses a method for finding the gradient of a curve graphically, by drawing a tangent to the curve at a particular point. Section 10.2 introduces a method for deriving an *equation* for the gradient from the equation of the curve itself; this method is known as differentiation. Differentiation is one of the branches of calculus (integration, the other major branch of calculus, is beyond the scope of this course), where the word calculus comes from the Latin for ‘a stone’ and relates to the use of stones for counting, or calculating. This chapter is about calculating rate of change.

10.1 Drawing tangents to curves

For a straight line, the gradient is the same at all points. However, the gradient of a curve varies from point to point. If you look at Figure 10.1 from left to right, you will see that the slope of the curve is initially gentle; then it gets steeper; then it reduces again. If this graph represents the way in which the height of our plant varies with time, this means that growth is initially slow, before increasing to a more rapid rate and then slowing again.

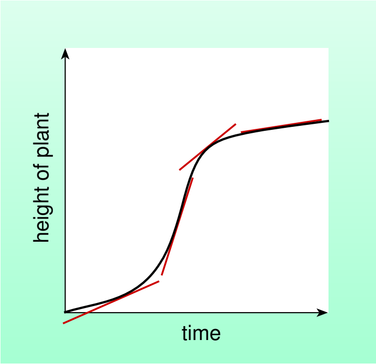


Figure 10.1: A curve, representing the growth of a hypothetical plant.

The straight lines drawn in red at various points on Figure 10.1 each have a slope that exactly matches the slope of the curve at the point at which it is drawn. These lines are called tangents, and the gradient of a curve at a point is defined to be the gradient of a tangent drawn at that point. The word tangent comes from the Latin ‘tangere’ which means ‘to touch’, and a tangent is a line which touches the curve but doesn’t cross it. Note that the use of the word ‘tangent’ here is different from its use in trigonometry (Chapter 6).

Figure 10.2 illustrates the fact that, at each point, there is only one line that touches a smooth curve without crossing it, so each point on the curve has a unique tangent and thus a unique gradient. This result is true for all points on all smooth curves.

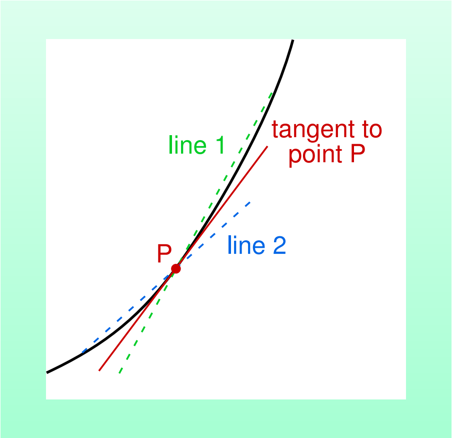


Figure 10.2: The tangent to a curve at a point P. Note that there is only one tangent at P. Line 1, with a gradient slightly greater than that of the tangent, and line 2, with a gradient slightly smaller than that of the tangent, both cross the curve.

A tangent is a straight line, so we can find its gradient using the method discussed in Chapter 5. Figure 10.3 is a graph of $y = x^2$ and tangents have been drawn at $x = 1$ and at $x = 3$.

Using the triangle drawn on the graph, the gradient of the tangent at $x = 3$ is

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{(15.0 - 9.0)}{(4.0 - 3.0)} = \frac{6.0}{1.0} = 6.0$$

Note that, because on this occasion x and y are variables without units, the gradient also has no units.

The gradient of the curve at a point is the same as that of the tangent touching the curve at that point, so we can say that the gradient of the curve at $x = 3$ is 6.0 to two significant figures.

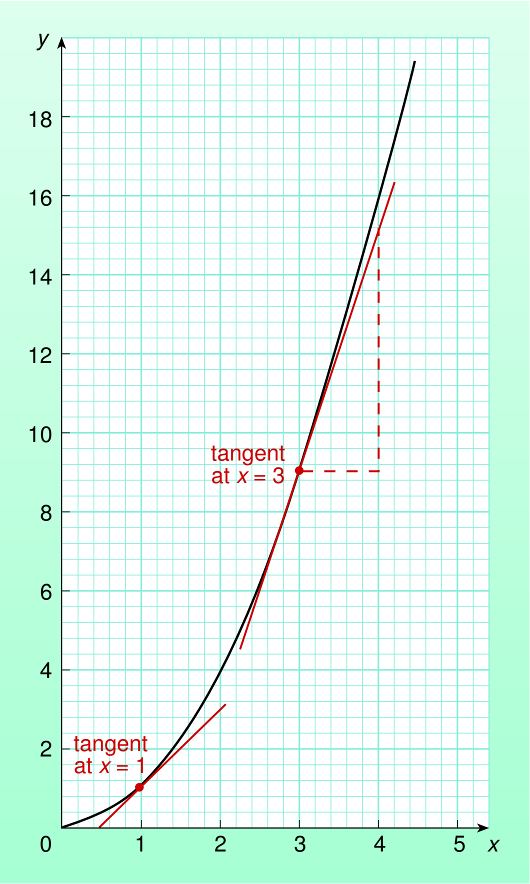


Figure 10.3: A graph of $y = x^2$

Question 10.1

(a) Find the gradient of $y = x^2$ at $x = 1$ by finding the gradient of the tangent which has been drawn to Figure 10.3 at $x = 1$. Answer

(b) Find the gradient of $y = x^2$ at $x = 2$ by drawing an additional tangent to the curve in Figure 10.3. Answer

Box 10.1 Rate of change of concentration in chemical reactions

As a chemical reaction involving substances in solution proceeds, the concentrations of the substances (called 'reactants' and 'products') vary with time.

Figure 10.4 shows the way in which the concentration of one of the products of a particular reaction increases with time. The product in this case is called a hypobromite ion.

To find the rate of change of concentration of hypobromite ions with time at any instant, we can draw a tangent to the curve and find its gradient. A tangent has been drawn to the curve in Figure 10.4 at 1500 s.

$$\begin{aligned}\text{The gradient of the tangent} &= \frac{2.40 \times 10^{-3}\ \text{mol dm}^{-3} - 1.16 \times 10^{-3}\ \text{mol dm}^{-3}}{3000\ \text{s} - 0\ \text{s}} \\ &= 4.13 \times 10^{-7}\ \text{mol dm}^{-3}\,\text{s}^{-1}\end{aligned}$$

So the rate of change of concentration of hypobromite ions with time at 1500 s is $4.13 \times 10^{-7}\ \text{mol dm}^{-3}\,\text{s}^{-1}$.

10.2 An introduction to differentiation

In answering Question 10.1 you probably realized that drawing tangents to curves is not a very accurate way of finding gradients. Using this method, the gradient of $y = x^2$ at $x = 2$ could reasonably be anything between 3.5 and 4.5, although the correct answer is exactly 4 (as you will discover in Section 10.2.1). Fortunately, when the equation of the curve is known (as it is in this case), differentiation gives us an exact method for finding the gradient, without even having to draw a graph.

10.2.1 The principles of differentiation

The reason why drawing a tangent to a curve is tricky is that, by definition, a tangent only goes through one point on the curve and this makes it difficult to draw a line with the correct gradient. Drawing a chord (a line between two points on the curve) and finding its gradient is very much easier.

The chord shown joining point P and point Q in Figure 10.5 (next page) has gradient $\dfrac{\Delta y}{\Delta x}$, where Δy is the difference between the y values of P and Q and Δx is the corresponding difference between x values (Δ, the Greek upper case delta, is used to indicate the change in a quantity, as discussed in Chapter 3).

As point Q moves along the curve towards P, passing through Q_1, Q_2 and Q_3, two things happen.

1. The values of Δx and Δy get smaller and smaller.
2. The gradient of the chord gets closer and closer in value to the gradient of the tangent at P.

If we reduce Δx all the way to zero, Δy will also be zero, making $\dfrac{\Delta y}{\Delta x}$ rather difficult to define, but we can make Δx as small as we like in order to get an accurate measurement of the gradient. This situation is described as a 'limit'; as Δx approaches zero, the approximation $\dfrac{\Delta y}{\Delta x}$ approaches ever closer to the exact gradient of the curve at the specified point. In this limit, $\dfrac{\Delta y}{\Delta x}$ is written as $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ where $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ (said as 'dee y by dee x') is called the derivative (or, strictly, the 'first derivative') of y with respect to x.

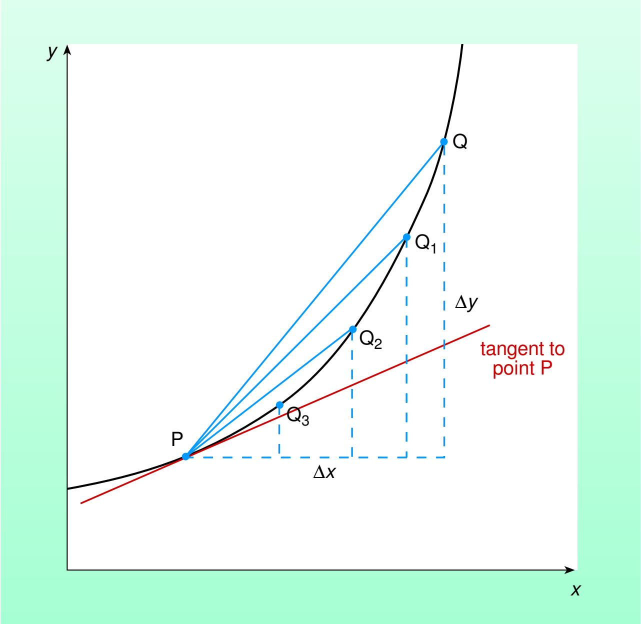


Figure 10.5: Finding the gradient of a curve at P.

Note that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ should be regarded as a single symbol. It does *not* mean a quantity $\mathrm{d}y$ divided by another quantity $\mathrm{d}x$, and the 'd's are not separate quantities so they cannot be cancelled: $\dfrac{\mathrm{d}y}{\mathrm{d}x} \neq \dfrac{y}{x}$.

Differentiation is simply the process of finding a derivative. Box 10.2 shows how this can be done from first principles for the example we have been considering, $y = x^2$. This box is included for interest only; you do not need to be able to differentiate from first principles. All you need to be able to do is to apply some very simple general rules (the first of which is discussed in Section 10.2.2, immediately after the box) that enable you to find the derivative directly from the original equation. It turns out that for $y = x^2$ we can say straight away that $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, so the gradient at $x = 1$ is $(2 \times 1) = 2$, the gradient at $x = 2$ is $(2 \times 2) = 4$, and the gradient at $x = 3$ is $(2 \times 3) = 6$; reassuringly these are the same results that we obtained earlier by drawing tangents to the curve, but now the answers are exact and we have found them without having to draw a graph.

Box 10.2 Differentiating $y = x^2$ from first principles

Consider the chord drawn between points P and Q on Figure 10.6. P could be any point on the curve, so its x and y values are related by the equation $y = x^2$.

The x value at Q is $(x + \Delta x)$ and the y value is $(y + \Delta y)$. Since point Q lies on the curve too, we can say

$$(y + \Delta y) = (x + \Delta x)^2$$

Multiplying out the bracket on the right-hand side, in the way discussed in Chapter 4, gives

$$y + \Delta y = x^2 + 2x\Delta x + (\Delta x)^2$$

Since $y = x^2$, we can subtract y from the left-hand side and x^2 from the right-hand side to give

$$\Delta y = 2x\Delta x + (\Delta x)^2$$

Dividing both sides by Δx gives

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

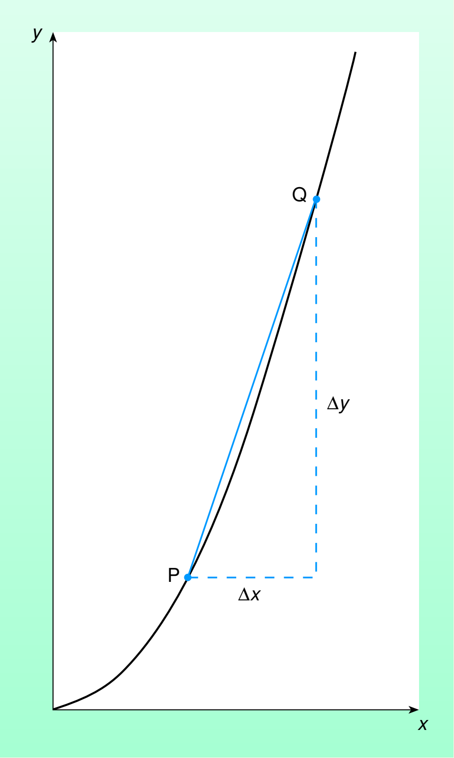


Figure 10.6: Points P and Q on the curve $y = x^2$.

In the limit as Δx approaches zero, the second term on the right-hand side will disappear, and $\frac{\Delta y}{\Delta x}$ will become equal to $\frac{dy}{dx}$, so we can say

$$\frac{dy}{dx} = 2x$$

10.2.2 Differentiation by rule

It was shown, in Box 10.2, that the derivative of $y = x^2$ with respect to x is $\frac{dy}{dx} = 2x$.

By similar methods, it can be shown that:

- the derivative of $y = 2\,x^2$ with respect to x is $\frac{dy}{dx} = 4x$;
- the derivative of $y = 3\,x^2$ with respect to x is $\frac{dy}{dx} = 6x$;
- the derivative of $y = 4\,x^2$ with respect to x is $\frac{dy}{dx} = 8x$.

or, more generally, the derivative of $y = C\,x^2$ with respect to x, where C is a constant, is

$$\frac{dy}{dx} = C \times 2x$$

Similarly, it can be shown that:

- the derivative of $y = C\,x^3$ with respect to x is $\dfrac{\mathrm{d}y}{\mathrm{d}x} = C \times 3x^2$;
- the derivative of $y = C\,x^4$ with respect to x is $\dfrac{\mathrm{d}y}{\mathrm{d}x} = C \times 4x^3$;
- the derivative of $y = C\,x^5$ with respect to x is $\dfrac{\mathrm{d}y}{\mathrm{d}x} = C \times 5x^4$.

These results can be summarized in the general rule

The derivative of $y = C\,x^n$ with respect to x is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = C\,n\,x^{n-1}$$

where C and n are constants.

Worked example 10.1

If $y = x^5$, what is $\frac{dy}{dx}$ and what is the gradient of a graph of $y = x^5$ at $x = 2$?

Answer

In this case $C = 1$ and $n = 5$, so $\frac{dy}{dx} = 1 \times 5x^4 = 5x^4$.

When $x = 2$, $\frac{dy}{dx} = 5 \times 2^4 = 5 \times 16 = 80$.

So at $x = 2$ the gradient of the graph is 80.

Worked example 10.2

If $y = 4x^3$, what is $\frac{dy}{dx}$ and what is the gradient of a graph of $y = 4x^3$ at $x = 3$?

Answer

In this case $C = 4$ and $n = 3$, so $\frac{dy}{dx} = 4 \times 3x^2 = 12x^2$.

When $x = 3$, $\frac{dy}{dx} = 12 \times 3^2 = 12 \times 9 = 108$.

So at $x = 3$ the gradient of the graph is 108.

Worked example 10.3 considers the application of the rule for differentiation in the special case when $n = 1$, and Worked example 10.4 considers what happens when

$n = 0$; you may like to think about what you expect the results to be.

Worked example 10.3

If $y = 4x$, what is $\dfrac{dy}{dx}$?

Answer

In this case $C = 4$ and $n = 1$, so $\dfrac{dy}{dx} = 4 \times 1x^{1-1} = 4x^0 = 4$, (since $x^0 = 1$ for all values of x, as discussed in Section 1.3.1).

Note that $y = 4x$ is a linear equation of the form $y = kx$, so the result of Worked example 10.3 should not have surprised you; differentiating an equation of the form $y = kx$ will always result in a derivative which is a constant. This constant is equal to the gradient, k, of a graph of y against x (as discussed in Section 5.3.1).

Worked example 10.4

If $y = 3$, what is $\dfrac{dy}{dx}$?

Answer

$y = 3$ can be written as $y = 3x^0$ (since $x^0 = 1$), so $C = 3$ and $n = 0$.

Thus $\dfrac{dy}{dx} = 3 \times 0 \times x^{-1} = 0$ (since multiplying anything by 0 gives 0).

Differentiating a constant always gives zero. This should not surprise you either, since the graph of $y = 3$ is a horizontal line and the gradient of a horizontal line is always zero.

Question 10.2

Differentiate the following with respect to x and in each case find the gradient of the graph of y against x at $x = 4$.

Note that the instruction 'to differentiate' simply requires you to find the derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(a) $y = x^4$ Answer

(b) $y = 5x$ Answer

(c) $y = 3x^2$ Answer

(d) $y = 5$ Answer

The rule for differentiation that we have been using applies for negative and fractional values of n too, as illustrated in Worked examples 10.5 and 10.6.

Worked example 10.5

Differentiate $y = \dfrac{3}{x}$ with respect to x.

Answer

$y = \dfrac{3}{x}$ can be written as $y = 3x^{-1}$ (see Section 1.3.1 for a reminder of the use of negative exponents), so $C = 3$ and $n = -1$.

Thus $\dfrac{dy}{dx} = 3 \times (-1)x^{-1-1} = -3x^{-2} = -\dfrac{3}{x^2}$

Worked example 10.6

Differentiate $y = \sqrt{x}$ with respect to x.

Answer

$y = \sqrt{x}$ can be written as $y = x^{1/2}$ (see Section 1.3.4), so $C = 1$ and $n = \frac{1}{2}$.

Thus $\dfrac{dy}{dx} = \dfrac{1}{2}x^{\frac{1}{2}-1} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2x^{1/2}} = \dfrac{1}{2\sqrt{x}}$.

Question 10.3

Differentiate the following with respect to x and in each case find the gradient of the graph of y against x at $x = 4$.

(a) $y = \dfrac{1}{\sqrt{x}}$ Answer

(b) $y = \dfrac{2}{x^2}$ Answer

10.2.3 Using different symbols and different notation

So far we have found derivatives only of y with respect to x. For example we differentiated $y = x^2$ and found that $\dfrac{dy}{dx} = 2x$. Note that y and $\dfrac{dy}{dx}$ are both functions of x; this means that the values of y and $\dfrac{dy}{dx}$ depend on the value of x. A derivative is sometimes called a 'derived function' because it is a function that has been derived from another function.

Functions in science are often expressed in terms of variables other than x and y. For example, we may know that as time, t, changes, the distance, s, of an object from a certain position varies according to the equation $s = 5t^2$. The graph of this function is illustrated in Figure 10.7.

The speed at which the object is moving is given by the rate of change of distance with time, so to find the object's speed we need to find the gradient of the graph shown in Figure 10.7, i.e. to differentiate s with respect to t.

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 5 \times 2t^{2-1} = 10\,t$$

Similarly, we know from Chapter 5 that the volume, V, of a gas at constant temperature is inversely proportional to its pressure, P,

$$V \propto \frac{1}{P} \text{ or } V = \frac{k}{P} = kP^{-1}$$

where k is a constant.

Differentiating V with respect to P gives

$$\frac{\mathrm{d}V}{\mathrm{d}P} = k \times (-1)P^{-1-1} = -kP^{-2} = -\frac{k}{P^2}$$

This expression gives the gradient of the graph shown in Figure 5.29.

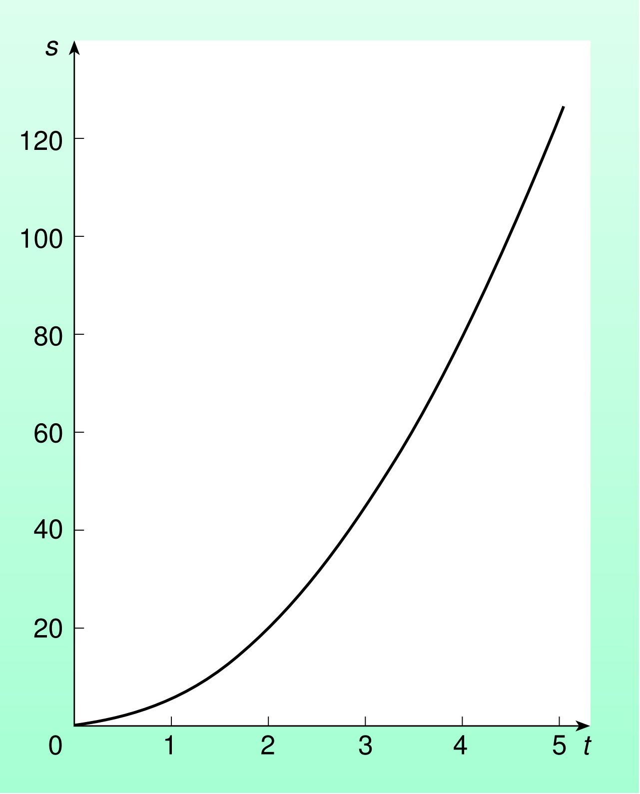


Figure 10.7: A graph of $s = 5t^2$

Question 10.4

(a) Differentiate $x = t^7$ with respect to t. Answer

(b) If $E = \dfrac{C}{r}$ where C is constant, what is $\dfrac{\mathrm{d}E}{\mathrm{d}r}$? Answer

An entirely different notation, called function or prime notation, is sometimes used for derivatives. This notation makes it very clear that both the expression being differentiated and its derivative are functions, and it identifies the variable on which the functions depend. In this notation, the function shown in Figure 10.7 would be written as $f(t) = 5t^2$ and its first derivative would be written as $f'(t) = 10t$. The term $f(t)$, usually said as 'f of t', does not mean f times t, but simply implies that f is a function of t. The term $f'(t)$ (said as 'f prime of t') is the first derivative of f with respect to t.

Unfortunately both $f'(x)$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ notation are in common use, as is a variation of the latter which writes $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(5t^2\right) = 10t$ for the derivative of $5t^2$ with respect to time. This course uses only $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ notation as discussed in the preceding sections, but you should be aware that other notations are also widely used. $f'(x)$ notation is relatively modern but $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ notation was invented by Gottfried Leibniz, one of the founders of calculus, and is known as Leibniz notation.

Yet another notation, less commonly used in modern times, writes $\dot{s}$ for the first derivative of $s = 5t^2$ with respect to t. This notation was first used by Newton, and the fact that we are left with such a plethora of notations for differentiation is a lasting reminder of the bitter dispute between Newton and Leibniz over which of them invented calculus (see Box 10.3).

Box 10.3 Newton and Leibniz: a story of reluctant publishers and letters 'lost in the post'

Sir Isaac Newton (1642–1727) and the German mathematician and philosopher Gottfried Wilhelm Leibniz (sometimes spelt Leibnitz) (1646–1716) both claimed to have invented calculus. It is probable that they developed the ideas independently; they certainly described their work in very different ways. Newton thought in terms of 'fluxions' whilst Leibniz used 'differences' (hence the word 'differentiation') and developed the $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ notation still in use today.

Leibniz published a paper about differentiation in 1684 and another about integration in 1686. Newton had problems getting his mathematical work into print; the publisher of his colleague Isaac Barrow's work had gone bankrupt and publishers were wary of mathematical works after this. Works written by Newton in 1669 and 1671 were not published until 1711 and 1736 respectively.

Another source of the controversy seems to have been the length of time it took for a letter to get from Newton in Cambridge to Leibniz in Paris. Newton's letter listed many of his results, and when Leibniz's reply took a long time to arrive, Newton assumed that Leibniz had spent six weeks refining his own work in the light of Newton's before replying. According to Leibniz the original letter had spent these six weeks on its way from Cambridge to Paris, and he had replied immediately he had received it.

It is beyond doubt that Newton accused Leibniz of plagiarism and that, despite the fact that both men were well respected within their lifetimes and famous afterwards, they ended their lives in acrimonious dispute with each other.

10.2.4 Differentiating sums

Suppose we need to differentiate $y = x^2 - 4x + 3$, the function shown in Figure 10.8, with respect to x.

It is possible to do this differentiation from first principles, as shown in Box 10.4. Once again, this box is included for interest only, as it turns out that it is possible to differentiate $y = x^2 - 4x + 3$ by the application of the rule already introduced, and another simple rule which is stated after the box. It would be possible to differentiate all functions from first principles, but it is a lot quicker simply to apply the rules!

It is shown in Box 10.4 that if

$$y = x^2 - 4x + 3 \tag{10.1}$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 4 \tag{10.2}$$

We could write $y = x^2 - 4x + 3$ as $y = u + v + w$ where $u = x^2$, $v = -4x$ and $w = 3$. We know (from the rule introduced in Section 10.2.2) that if $u = x^2$, then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \tag{10.3}$$

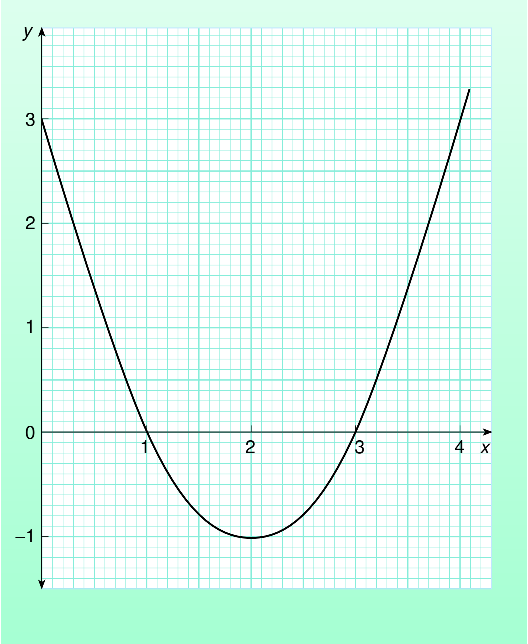


Figure 10.8: A graph of $y = x^2 - 4x + 3$.

if $v = -4x$, then

$$\frac{dv}{dx} = -4 \tag{10.4}$$

and if $w = 3$, then

$$\frac{dw}{dx} = 0. \tag{10.5}$$

Comparing Equation 10.2 with Equations 10.3, 10.4 and 10.5 shows that

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}$$

This rule is a general one, in other words:

> The derivative of the sum of a number of functions is equal to the sum of the derivatives of these functions. If
>
> $$y = u + v + w$$
>
> then
>
> $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}$$

Worked example 10.7

Differentiate $x = t^5 + 6t^3$ with respect to t.

Answer

Differentiating each of the terms separately gives

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= (1 \times 5t^{5-1}) + (6 \times 3t^{3-1}) \\ &= 5t^4 + 18t^2\end{aligned}$$

Question 10.5 Answer

Differentiate $z = 4y^2 + y$ with respect to y.

10.2.5 Second derivatives

$\dfrac{\mathrm{d}y}{\mathrm{d}x}$ gives the gradient of a graph of y against x. It is often also useful to know the rate of change of the *gradient* with respect to x, i.e. to differentiate again with respect to x to find the derivative of the derivative. Such a quantity is referred to as the second derivative of y with respect to x and it is written $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ (said as 'dee-2-y by dee-x-squared') or $f''(x)$ (said as 'f double prime of x') in function notation.

Consider again the example used in Section 10.2.4.

We had

$$y = x^2 - 4x + 3$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 4$$

$\frac{\mathrm{d}y}{\mathrm{d}x}$ is itself a function of x and differentiating again gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$$

The graphs of y against x, $\frac{\mathrm{d}y}{\mathrm{d}x}$ against x and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ against x for this example are shown in Figure 10.10. The graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ against x shows how the *gradient* of the graph of y against x varies with x, and the graph of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ against x shows how the gradient of the graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ against x varies with x. In this particular case, the graph of y against x (Figure 10.10a) is a parabola (as discussed in Section 5.4). Note that this graph is horizontal at $x = 2$; at this point its gradient is zero. It should not surprise you, therefore, that the graph of $\frac{dy}{dx}$ against x (Figure 10.10b) has $\frac{dy}{dx} = 0$ at $x = 2$. Similarly, the graph of $\frac{dy}{dx}$ against x is a straight line of gradient 2, so the fact that $\frac{d^2y}{dx^2}$ has a constant value of 2 (see Figure 10.10c) should not surprise you.

Question 10.6

Find the first and second derivatives of:

(a) $x = 2t^3 + 4t^2 - 2t + 3$ with respect to t; Answer

(b) $z = \frac{2}{y}$ with respect to y. Answer

Box 10.5 considers an application of differentiation to science, in this case the motion of an object falling because of the action of gravity. Note that the variables are now real physical quantities, so they have units attached to them.

Box 10.5 Objects falling under gravity

Suppose that an object is dropped from the Clifton Suspension Bridge, which crosses the River Avon as it flows through a gorge near Bristol. The bridge is 75 m above the river, as illustrated in Figure 10.11.

If we assume that the object starts from rest (i.e. it is dropped not thrown from the bridge) then the distance, s, that it has travelled downwards from the bridge in a time t is given by the equation

$$s = \frac{1}{2}gt^2 \tag{10.7}$$

where g is the magnitude of the acceleration due to gravity, which we can take to be $9.81\ \mathrm{m\,s^{-2}}$.

We can differentiate Equation 10.7 twice in order to find out more about the way the object's speed changes as it falls. However, first let's find the time taken for the object to reach the river. Rearranging Equation 10.7 to make t^2 the subject gives

$$t^2 = \frac{2s}{g}$$

Taking the square root of both sides (recognizing that t is a period of time so we are only interested in the positive square root) gives

$$t = \sqrt{\frac{2s}{g}}$$

Thus, when $s = 75$ m,

$$t = \sqrt{\frac{2 \times 75\ \mathrm{m}}{9.81\ \mathrm{m\,s^{-2}}}} = \sqrt{15.29\ \mathrm{s}^2} = 3.91\ \mathrm{s} = 3.9\ \mathrm{s}\ \text{to two significant figures.}$$

So the object takes 3.9 seconds to hit the water.

The object starts from rest, but will be travelling quite fast when it hits the water. How fast? To find speed we need to find the rate of change of distance, i.e. to differentiate Equation 10.7 with respect to t.

The speed v is then

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{2}g \times 2t = gt$$

This implies that speed is proportional to time; the speed is zero as the object is dropped but then it increases in a linear way as time increases.

Since it takes 3.9 s for the object to hit the water (or 3.91 s, working to three significant figures to avoid rounding errors), its speed as it hits the water is

$$v = 9.81\ \mathrm{m\,s^{-2}} \times 3.91\ \mathrm{s} = 38\ \mathrm{m\,s^{-1}}\ \text{to two significant figures.}$$

Differentiating Equation 10.7 for a second time tells us the rate at which the object's *speed* is changing. This is the object's acceleration, a

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = g = 9.81\ \mathrm{m\,s^{-2}}$$

Thus the object is accelerating at $9.81\ \mathrm{m\,s^{-2}}$ (the acceleration due to gravity) as you might have expected. The fact that the final answer is reasonable provides a useful check of Equation 10.7 (which was *assumed* to be the correct equation from which to start). Note that the acceleration is constant all the time the object is falling, and the fact that acceleration is positive is consistent with the observed fact that speed increases as the object falls.

Figure 10.12 shows the variation of the object's distance s from the bridge, speed v, and acceleration in a downwards direction a, with increasing time. Note that the gradient of the first graph (s against t) leads to the second graph (v against t) and that the gradient of the second graph (v against t) leads to the final graph (a against t).

10.3 Differentiating exponential functions

Chapter 5 introduced graphs of exponential growth, such as $n = n_0\,\mathrm{e}^{at}$ (see Figure 5.36) and graphs of exponential decay, such as $N = N_0\,\mathrm{e}^{-\lambda t}$ (see Figure 5.35), where e is the number whose value is 2.718 to four significant figures. In general a function of the type $y = C\,\mathrm{e}^{kx}$, where x and y are variables and C and k are constants, is called an exponential function. Finding the gradient of exponential functions reveals another reason why e is such a special number.

Figure 10.13 is a graph of the simplest imaginable exponential function; in this case $C = 1$ and $k = 1$, so $y = \mathrm{e}^{x}$. Tangents have been drawn to the curve in Figure 10.13 at $y = 1$, $y = 5$ and $y = 10$.

Question

Use the tangents that have been drawn on Figure 10.13 to find the gradient of the graph of $y = e^x$ at $y = 1$, $y = 5$ and $y = 10$. You should work to two significant figures in each case.

Answer

The gradient of the tangent drawn at $y = 1$ is

$$\text{gradient} = \frac{(2.0 - 0.0)}{(1.0 - (-1.0))} = \frac{2.0}{2.0} = 1.0$$

The gradient of the tangent drawn at $y = 5$ is

$$\text{gradient} = \frac{(9.2 - 4.2)}{(2.5 - 1.5)} = \frac{5.0}{1.0} = 5.0$$

The gradient of the tangent drawn at $y = 10$ is

$$\text{gradient} = \frac{(17.0 - 7.0)}{(3.0 - 2.0)} = \frac{10.0}{1.0} = 10$$

In each case, to two significant figures, the gradient of the tangent (and thus of the graph itself) is equal to the value of y at the point where the tangent was drawn.

Question

Predict the gradient of a tangent drawn to the curve in Figure 10.13 at $y = 2$.

Answer

It seems likely that a tangent drawn at $y = 2$ will have a gradient of 2 too. It turns out that this is indeed the case.

The rule that has emerged from this sequence is generally true; the gradient of a graph of $y = \mathrm{e}^{x}$ at a particular point is equal to the value of y at that point, i.e. for $y = \mathrm{e}^{x}$, the derivative of y with respect to x is equal to y itself:

If $y = \mathrm{e}^{x}$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y$

or, put another way, if $y = \mathrm{e}^{x}$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{x}$.

$\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is only *equal* to y for this one specific exponential function. However, more generally,

If $y = C\,\mathrm{e}^{kx}$, where C and k are constants, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = Ck\,\mathrm{e}^{kx}$.

Since $y = C\,\mathrm{e}^{kx}$ this means that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = ky$ i.e. $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is *proportional* to y.

This rule is the final rule for differentiation given in this course, and its use is illustrated in Worked examples 10.8 and 10.9. The fact that $\dfrac{dy}{dx}$ is proportional to y for all functions of the form $y = C\,e^{kx}$ also explains the shape of graphs of exponential growth and decay, and is the reason why e is such a special number.

Worked example 10.8

If $y = e^{3x}$ what is $\dfrac{dy}{dx}$? Express your answer (a) in terms of x, (b) in terms of y.

Answer

(a) $C = 1$ and $k = 3$ in this case, so $\dfrac{dy}{dx} = 1 \times 3\,e^{3x} = 3\,e^{3x}$.

(b) Alternatively, we could write $\dfrac{dy}{dx} = 3\,y$.

Worked example 10.9

Find the first and second derivatives of $z = 3\,e^{-2t}$ with respect to t.

Answer

$C = 3$ and $k = -2$ in this case, so $\dfrac{dz}{dt} = 3 \times -2e^{-2t} = -6\,e^{-2t}$.

This could also be written as $\dfrac{dz}{dt} = -2z$.

Differentiating again gives

$$\frac{d^2z}{dt^2} = -6 \times -2\,e^{-2t} = 12\,e^{-2t}$$

This could also be written as $\dfrac{d^2z}{dt^2} = 4z$.

Question 10.7

(a) Differentiate $y = 2\,e^x$ with respect to x. Answer

(b) Differentiate $z = e^{t/2}$ with respect to t. Answer

Since $\dfrac{dy}{dx}$ is proportional to y for all exponential functions we can use exponential functions to describe situations in which the rate of change of some quantity at an instant is proportional to the actual value of that quantity at the same instant.

Radioactive decay, first described in Chapter 5 and further discussed in Box 10.6, is an example of a situation like this and this is why it can be described by an exponential function.

Box 10.6 Radioactive decay revisited

As previously discussed, radioactive decay is an inherently random process; we can never know when a particular nucleus is going to decay. However, if the sample contains a large number of radioactive nuclei, the *average* rate at which the nuclei decay will be proportional to the number of radioactive nuclei remaining. It is this proportionality which leads to the process being an exponential one, in this case an exponential *decay*.

As discussed in Chapter 5 and Chapter 7, we can describe the process by the equation $N = N_0\,\mathrm{e}^{-\lambda t}$ where λ is a positive constant called the decay constant, N_0 is the initial number of radioactive nuclei and N is the number remaining after time t. The constant λ is related to the half-life, $t_{1/2}$ by the equation $t_{1/2} = \dfrac{\ln 2}{\lambda}$ (see Box 7.5). Differentiating $N = N_0\,\mathrm{e}^{-\lambda t}$ with respect to t gives

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -N_0\lambda\,\mathrm{e}^{-\lambda t} = -\lambda\,N \text{ (since } N = N_0\,\mathrm{e}^{-\lambda t}\text{)}.$$

In this case the gradient is negative as you would expect, since the number of radioactive nuclei remaining is reducing with increasing time. The larger the value of N, the faster the rate of decay, as shown in Figure 10.14.

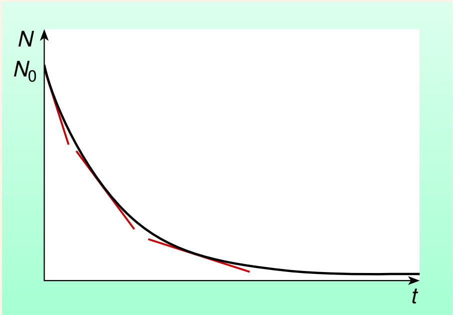


Figure 10.14: Radioactive decay can be described by the equation $N = N_0\,\mathrm{e}^{-\lambda t}$, so $\dfrac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$. Tangents to the curve are shown in red.

10.4 Learning outcomes for Chapter 10

After completing your work on this chapter you should be able to:

10.1 demonstrate understanding of the terms emboldened in the text;

10.2 find the gradient of a curve at a particular point by means of drawing a tangent to the curve at that point;

10.3 demonstrate understanding of the fact that the derivative of a function gives the gradient of the corresponding graph;

10.4 demonstrate understanding of the fact that the second derivative of a function is obtained by differentiating twice;

10.5 differentiate functions of the form $y = C\,x^n$;

10.6 differentiate simple sums of functions;

10.7 differentiate exponential functions of the form $y = C\,\mathrm{e}^{kx}$;

10.8 demonstrate understanding of the fact that, for exponential functions, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is proportional to y.

A Resolving vectors

The letter v has been used throughout the course to represent speed; but why v rather than s? The letter v reminds us of the word 'velocity' which, in everyday speech, is used interchangeably with speed. However, in science the two words have subtly different meanings. Velocity is an example of a vector, a quantity that has direction as well as magnitude (size). In contrast, speed is a scalar quantity; it has magnitude only.

Question

In terms of a strict interpretation of vector and scalar quantities, what is wrong with the statement 'the car has a velocity of 50 km hour^{-1}'?

Answer

No direction has been given, so this is a scalar quantity, i.e. the speed of the car. To turn it into a vector we would need to say, for example, that 'the car has a velocity of 50 km hour^{-1} due north'.

Another example of the difference between speed and velocity comes when considering an object orbiting another object at constant speed. Consider, for example, the Earth orbiting the Sun at about 30 $\mathrm{km\,s^{-1}}$ (as discussed in Box 3.1). The Earth's speed relative to the Sun is approximately constant, but its direction of movement is constantly changing, so its velocity is constantly changing too.

The quantities considered elsewhere in this course have been almost exclusively scalars (mass, temperature, energy, magnitude of acceleration) but velocity is not the only scientific quantity to be a vector, by a very long way. Other such quantities include, force, weight and acceleration.

A vector may be represented diagrammatically by an arrow, the length of which specifies the vector's magnitude, and the direction of which is the same as the vector's. By convention, vectors are printed as bold symbols, e.g. $\boldsymbol{v}$, while the magnitude of the vector is written normally, e.g. v. Handwritten vector symbols should be written with a wavy underline, as shown in Figure A.1.

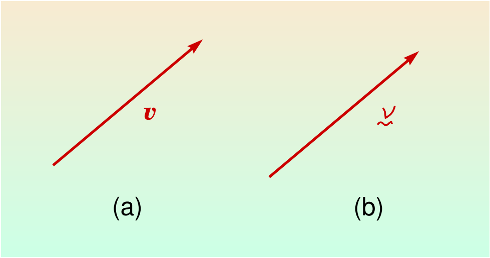


Figure A.1: Representing a vector: (a) in printed text; (b) by hand.

To specify a vector fully, both its magnitude (which is always positive) and its direction must be stated, e.g. '$\boldsymbol{F}$ is a force of 10 N acting vertically downwards'. The magnitude of $\boldsymbol{F}$ may be written as

$$F = |\boldsymbol{F}| = 10 \text{ N}$$

The vertical lines drawn either side of the $\boldsymbol{F}$ provide an alternative way of indicating the modulus (magnitude) of the vector.

Adding vector quantities together is not as straightforward as adding scalar quantities, since both magnitude and direction need to be taken into account. Fortunately the trigonometry from Chapter 6 comes to our aid.

Imagine an object being acted on by the two forces shown in Figure A.2. You want to know the overall effect; what is the total force acting on the object as a result of $\boldsymbol{a}$ and $\boldsymbol{b}$? It is not immediately obvious how to proceed since the two forces have different sizes *and* are acting in different directions.

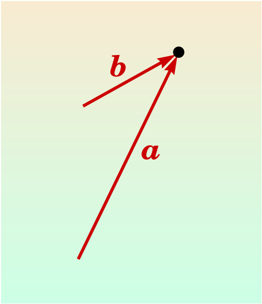


Figure A.2: Two forces $\boldsymbol{a}$ and $\boldsymbol{b}$ acting on an object.

One way forward is to resolve each vector into components; any two dimensional vector (such as one drawn on the page of a book, as here) can be characterized by its components along two perpendicular axes. Figure A.3 shows the components of the vector $\boldsymbol{a}$ along two axes x and y. Note that the components a_x and a_y are scalar quantities.

We can use trigonometry to find a_x and a_y.

Since $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$

we can say that $\cos\theta = \dfrac{a_x}{a}$, thus $a_x = a\cos\theta$.

Similarly $\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$,

so we can say that $\sin\theta = \dfrac{a_y}{a}$, thus $a_y = a\sin\theta$.

If $\boldsymbol{a}$ has magnitude $a = 6.0$ N and acts at 60° to the x-axis, we can say

$$a_x = 6.0\text{ N} \times \cos 60^\circ \qquad a_y = 6.0\text{ N} \times \sin 60^\circ$$
$$= 3.0\text{ N} \qquad\qquad = 5.2\text{ N}$$

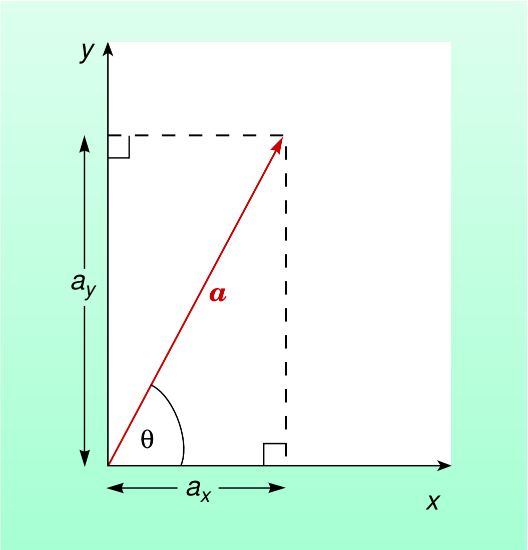


Figure A.3: The x- and y-components of $\boldsymbol{a}$.

Similarly, if $\boldsymbol{b}$ has magnitude $b = 2.8$ N and acts at 25° to the x-axis, we can say

$$b_x = 2.8\ \text{N} \times \cos 25^\circ = 2.5\ \text{N} \qquad b_y = 2.8\ \text{N} \times \sin 25^\circ = 1.2\ \text{N}$$

We can find the x-component of the resultant force $\boldsymbol{c}$, by *adding* the x-components of $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$c_x = a_x + b_x = 3.0\ \text{N} + 2.5\ \text{N} = 5.5\ \text{N}$$

Similarly, the y-component of $\boldsymbol{c}$ is given by

$$c_y = a_y + b_y = 5.2\ \text{N} + 1.2\ \text{N} = 6.4\ \text{N}$$

The resultant force $\boldsymbol{c}$ is shown in Figure A.4.

We can use Pythagoras' Theorem to find the magnitude c, so

$$\begin{aligned} c^2 &= c_x^2 + c_y^2 \\ c &= \sqrt{c_x^2 + c_y^2} \\ &= \sqrt{(5.5\ \text{N})^2 + (6.4\ \text{N})^2} \\ &= 8.4\ \text{N} \end{aligned}$$

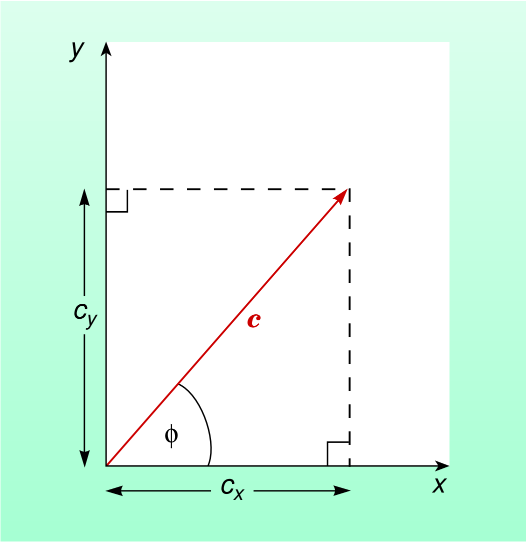


Figure A.4: Finding $\boldsymbol{c}$ from its x- and y-components.

And since $\tan\phi = \dfrac{\text{opp}}{\text{adj}} = \dfrac{c_y}{c_x}$ we can find the angle between $\boldsymbol{c}$ and the x-axis, which gives us the direction in which the force acts:

$$\tan\phi = \frac{6.4\ \text{N}}{5.5\ \text{N}} = 1.1636$$

Thus $\phi = \tan^{-1}(1.1636) = 49^\circ$ to two significant figures.

So the resultant force $\boldsymbol{c}$ has a magnitude of 8.4 N and acts at an angle of 49° to the horizontal axis.

Question A.1 Answer

Find the x- and y-components of the vector $\boldsymbol{v}$ shown in Figure A.5. The vector has a magnitude of 8.6 m s^{-1} and acts at an angle, α, of 42° to the x-axis.

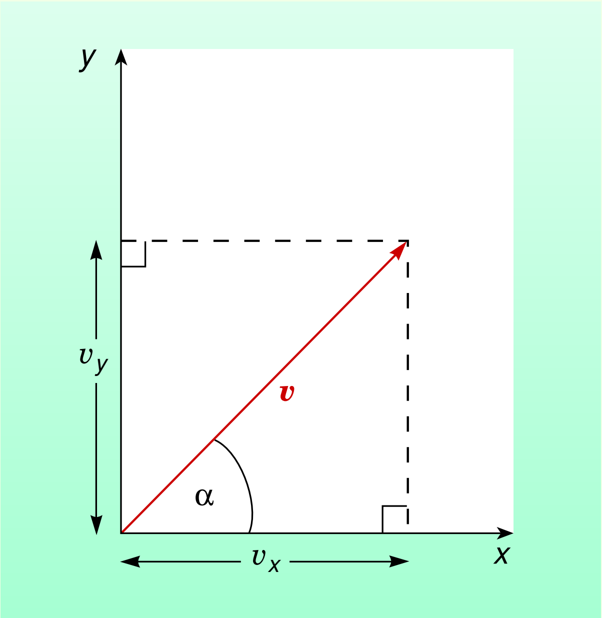


Figure A.5

Question A.2 Answer

Find the magnitude and direction of the vector $\boldsymbol{F}$ shown in Figure A.6.

$F_x = 4.0$ N and $F_y = 3.0$ N.

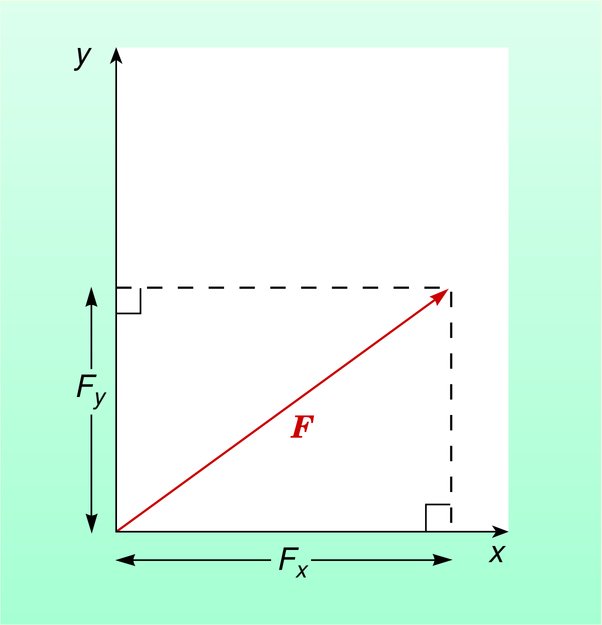


Figure A.6

Glossary

absolute-value The absolute value of a number is the number given without its + or – sign.

accurate Description of a set of measurements for which the systematic uncertainty is small. Compare with precise.

acute-angle An angle of less than 90°.

addition rule for probabilities A rule stating that if several possible outcomes are mutually exclusive, the probability of one or other of these outcomes occurring is found by adding their individual probabilities.

adjacent (trigonometry) The side other than the hypotenuse which is next to a particular angle in a right-angled triangle.

algebra The process of using symbols, usually letters, to represent quantities and the relationships between them.

alternative hypothesis The logical 'mirror image' of the null hypothesis proposed at the start of a statistical hypothesis test (e.g. that the means of two populations are not identical, $\mu_1 \neq \mu_2$).

arc A portion of a curve, particularly a portion of the circumference of a circle.

arccosine See inverse cosine.

arcsec An abbreviation for 'second of arc'. A 60th part of a minute of arc i.e. a 3600th part of a degree (of arc).

arcsine See inverse sine.

arctangent See inverse tangent.

arithmetic mean Measure of the average of a set of numbers. For a set of n measurements of a quantity x, the arithmetic mean $\overline{x}$ (often abbreviated to 'the mean') is defined as the sum of all the measurements divided by the total number of measurements:

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

See also the true mean.

arithmetic operations The operations of addition, subtraction, multiplication and division.

axis (of a graph) A horizontal or vertical reference line which carries a set of divisions. In the case of a bar chart the divisions may be a list of categories.

In the case of a graph the divisions indicate a linear or logarithmic scale, and are used to locate points on the graph.

bar chart A diagrammatic method of presenting data grouped into discrete categories. The categories are listed along one axis (usually the horizontal axis), and each category is represented by a bar (usually vertical). The bars are separated by gaps, and their height (or length) is directly proportional to the number or percentage of things or events in each category. Compare with histogram

base number When using exponents, the quantity that is raised to a power, e.g. 5 is the base in the statement $5 \times 5 \times 5 = 5^3$ and a is the base in the statement $a^3 \times a^4 = a^7$.

best-fit line A line (usually a straight line) drawn on a graph and chosen to be the best representation of the data as a whole. A best-fit line need not necessarily go through any of the data points (although it will typically go through some of them), and should be drawn in such a way that there are approximately the same number of data points above and below the line.

calculus The branch of mathematics which includes differentiation and integration.

cancellation The process of dividing both the numerator and denominator of a fraction by the same quantity. With numbers it may be quicker to use cancellation than to work out the value of the numerator and denominator separately, e.g.

$$\frac{5 \times \cancel{13}}{\cancel{13} \times 8} = \frac{5}{8}$$

Cancellation is also useful in simplifying algebraic expressions or units, e.g.

$$\frac{\cancel{a}bc^2}{2\cancel{a}d} = \frac{bc^2}{2d}$$

$$\frac{1\ \mathrm{N}\,\cancel{\mathrm{m}}}{1\ \mathrm{kg} \times 1\ \cancel{\mathrm{m}}} = \frac{1\ \cancel{\mathrm{kg}}\,\mathrm{m\,s^{-2}}}{1\ \cancel{\mathrm{kg}}} = 1\ \mathrm{m\,s^{-2}}$$

categorical level A level of measurement in which the data comprise distinct non-overlapping classes that cannot logically be ranked (e.g. presence versus absence, male versus female). See also ordinal level, interval level.

centi A prefix, used with units, to denote hundredths, and indicated by the symbol c. Thus one centimetre, denoted 1 cm, is the hundredth part of a metre. Centi is not one of the recognized submultiples in the system of SI units, but is nevertheless in common use, especially in association with units of length and volume.

χ^2 test (chi-squared test) A statistical hypothesis test used to determine whether there is a statistically significant association between two categorical level variables.

chord A line drawn between two points on a curve.

common denominator The same number or term occurring as the denominator of two or more fractions. For example, the numerical fractions $\frac{5}{16}$ and $\frac{7}{16}$ have the common denominator 16. It is often necessary to use equivalent fractions in order to find common denominators: for example $\frac{2}{5}$ ($= \frac{6}{15} = \frac{12}{30}$) and $\frac{8}{15}$ ($= \frac{16}{30}$) have common denominators 15 and 30 (as well as many other numbers). The algebraic fractions $\frac{a}{b}$ and $\frac{c}{d}$ have the common denominator $b \times d$.

common logarithm See logarithm to base 10.

commutative An operation for which the result is unchanged if the order of terms is reversed is described as commutative. Only two of the arithmetic operations are commutative: addition ($a + b = b + a$) and multiplication ($a \times b = b \times a$).

complex number A number of the form $n + mi$, where n is any real number, m is any non-zero real number, and $i = \sqrt{-1}$.

component (of a vector) The component of a vector along a chosen axis is obtained by drawing a line from the head of the arrow representing the vector onto the axis, such that the line meets the axis in a right angle. For example, the x-component of a vector $\boldsymbol{a}$ is $a_x = a \cos\theta$ where a is the magnitude of the vector and θ is the angle between the x-axis and the direction of the vector.

concentric Two circles are described as being concentric if they have the same centre.

constant of proportionality The constant factor that is required to turn a proportionality into an equation. The direct proportionality of $y \propto x$ can be written as $y = kx$, where k is the constant of proportionality.

conversion factor The number by which one needs to divide or multiply in order to convert from one unit to another.

correlation Two variables at ordinal level or interval level are said to be correlated if, as the value of one variable increases, the value of the second variable either increases (i.e. positive correlation) or decreases (i.e. negative correlation). If the values of the two variables increase precisely in step with one another, the positive correlation can be described as 'perfect'. In a 'perfect' negative correlation, the value of one variable decreases precisely as the other increases. Correlations may or may not be statistically significant.

correlation coefficient The correlation coefficient (r) of a 'perfect' positive correlation is +1, while that of a 'perfect' negative correlation is −1. When there is complete lack of correlation between two variables, $r = 0$. For a positive correlation that is less than 'perfect', $1 > r > 0$. For a negative correlation that is less than 'perfect', $0 > r > -1$.

cosine The cosine of an angle θ in a right-angled triangle is defined by

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

where 'adjacent' is the length of the side adjacent to θ and 'hypotenuse' is the length of the hypotenuse.

critical value At a particular number of degrees of freedom (in many statistical hypothesis tests), the critical value is the most extreme (usually the largest, but in some statistical tests the smallest) value that the test statistic is expected to have for a particular significance level.

deci Prefix, used with units, to denote tenths, and indicated by the symbol d. Thus one decibel, denoted 1 dB, is equal to one tenth of a bel. Deci is not one of the recognized submultiples in the system of SI units, but is commonly used in certain areas: for example the concentration of a chemical dissolved in a solvent is often expressed in units of moles per decimetre cubed (mol dm^{-3}).

decimal notation Method of representing numbers, according to which the integral and fractional parts of a number are separated by a decimal point. The decimal point is written as a full stop, with the integral part of the number to the left of it. The first digit after the decimal point indicates the number of tenths, the second indicates the number of hundredths, the third the number of thousandths, etc.

decimal places See places of decimals.

degree (of arc) A 360th of a complete revolution.

degree-Celsius An everyday unit of temperature, given the symbol °C. Pure water freezes at 0 °C and boils at 100 °C. Temperatures may be converted from degrees Celsius to the SI unit of temperature, kelvin, using the word equation (temperature in kelvin) = (temperature in degrees Celsius) + 273.15

degrees of freedom A device used in many statistical hypothesis tests to allow for the fact that the more data that are collected, the more scope there is for the test statistic to deviate from the value expected (generally, zero) if the null hypothesis were true.

denominator The number or term on the bottom of a fraction. For example, in the fraction $\frac{1}{2\pi}$, the denominator is 2π; in the fraction $\frac{mn}{pq}$, the denominator is pq. See also: numerator.

dependent variable A quantity whose value is determined by the value of one or more other variables. On a graph, the dependent variable is, by convention, plotted along the vertical axis. Compare with: independent variable.

derivative The derivative (or derived function) of a function $f(x)$ with respect to x is another function of x that is equal to the rate of change of $f(x)$ with respect to x. Its value at any given value of x is equal to the ratio $\frac{\Delta f}{\Delta x}$ in the limit as Δx becomes very small, and is usually written as $\frac{\mathrm{d}f}{\mathrm{d}x}$ or $f'(x)$. The value of $\frac{\mathrm{d}f}{\mathrm{d}x}$ at each value of x is also equal to the gradient of the graph of f plotted against x at that value of x. A derivative of the type is sometimes called the first derivative to distinguish it from the second derivative of the function.

derived function See derivative.

differentiation A mathematical process that enables the derivative of a function to be determined.

directly proportional (quantities) Two quantities x and y are said to be directly proportional to each other if multiplying (or dividing) x by a certain amount automatically results in y being multiplied (or divided) by the same amount. Direct proportionality between x and y is indicated by writing $y \propto x$. The direct proportionality can also be written as an equation of form $y = kx$, where k is a constant called the constant of proportionality. A graph in which y is plotted against x will be a straight line with gradient equal to k. See also inversely proportional.

elimination A method of combining two or more equations by eliminating variables that are common to them.

equation An expression containing an equals sign. What is written on one side of the equation must always be equal to what is written on the other side.

equivalent fractions Fractions that have the same value, e.g. $\frac{2}{3}, \frac{4}{6}, \frac{8}{12}, \frac{20}{30}$, etc.

estimated standard deviation of a population The best estimate that can be made for the standard deviation of some quantity for a whole population. This estimate is usually set equal to s_{n-1}, which is calculated from measurements of the quantity made on an unbiased sample drawn from the population. If the sample consists of n members and the quantity x is measured once for each member, then

$$s_{n-1} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

where $\overline{x}$ is the arithmetic mean of the measurements. The symbol σ_{n-1} is also widely used (especially on calculators) as an equivalent to s_{n-1}.

evaluate An instruction to work out the value of an expression.

exponent When raising quantities to powers, the number to which a quantity is raised, e.g. in the term 2^3, the exponent is 3.

exponential decay Decay in which the time taken for a quantity to fall to half its original value is always the same; this time is known as the half-life. A quantity N with an initial value of N_0 at time $t = 0$ decays exponentially if $N = N_0 \mathrm{e}^{-\lambda t}$, where λ is a constant known as the decay constant.

exponential function A function of the type $y = C\mathrm{e}^{kx}$ where C and k are constants. A function of this type has the property that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is proportional to y.

exponential growth Growth in which the quantity being measured increases by a constant factor in any given time interval. A quantity n with a starting value of n_0 at time $t = 0$ grows exponentially if $n = n_0\mathrm{e}^{at}$, where a is a positive constant.

expression A combination of variables (such as $a_x t$ or $u_x + a_x t$). Unlike an equation, an expression is unlikely to contain an equals sign.

extrapolation The process of extending a graph beyond the highest or lowest data points in order to find the values of one or both of the plotted quantities outside the original range within which data were obtained.

factor A term which when multiplied to other terms results in the original expression, so 6 and 4 are factors of 24 and $(a-3)$ and $(a+5)$ are factors of $a^2+2a-15$.

factorize To find the factors of an expression.

first derivative See derivative.

formula A rule expressed in algebraic symbols.

fraction A number expressed in the form of one integer divided by another, e.g. $\frac{1}{4}$; $\frac{3}{8}$; $\frac{21}{13}$. One algebraic term divided by another may also be described as a fraction. See also: improper fraction, mixed number, equivalent fractions, numerator and denominator.

function If the value of a variable f depends on the value of another variable x, then f is said to be a function of x and is written as $f(x)$. In general, there is only one value of $f(x)$ for each value of x.

gradient (of a graph) The slope of a line on a graph. The gradient is a measure of how rapidly the quantity plotted on the vertical axis changes in response to a change in the quantity plotted on the horizontal axis. If the graph is a straight line, then the gradient is the same at all points on the line and may be calculated by dividing the vertical 'rise' between any two points on the line by the horizontal 'run' between the same two points. If the graph is a curved line, the gradient at any point on the curve is defined by the gradient of the tangent to the curve at that point. See also: derivative.

graph A method of illustrating the relationship between two variable quantities by plotting the measured values of one of the quantities using a linear or logarithmic scale along a horizontal axis, and the measured values of the other quantity using a linear or logarithmic scale along a vertical axis. See also: dependent variable, independent variable, sketch graph.

half-life The time taken for half the nuclei in a radioactive sample to decay. See also exponential decay.

histogram A diagrammatic method of presenting data, in which the horizontal axis is divided into (usually equal) intervals of a continuously variable quantity. Rectangles of width equal to the interval have a height scaled to show the value of the quantity plotted on the vertical axis that applies at the particular interval. For example, the intervals could be the months in the year and the vertical axis could represent the mean (monthly) rainfall in millimetres. Compare with bar chart.

hyperbola A curve, part of which may be obtained by plotting inversely proportional quantities against each other on a .

hypotenuse The side opposite to the right-angle in a right-angled triangle.

hypothesis A plausible idea tentatively put forward to explain an observation. Traditionally, a hypothesis is tested by making predictions that would follow if the hypothesis is correct. If these predictions are borne out by experiment or further observation, then this lends weight to the hypothesis *but does not prove it to be correct*. If the predictions are not borne out, then the hypothesis is either rejected or modified.

imaginary number A number of the form mi, where m is any non-zero real number and $i = \sqrt{-1}$.

improper fraction A fraction in which the numerator is greater than the denominator, e.g. $\frac{12}{7}$. An improper fraction may also be written as a mixed number.

independent variable The quantity in an experiment or mathematical manipulation whose value(s) can be chosen at will within a given range. On a graph, the independent variable, is by convention, plotted along the horizontal axis. Compare with dependent variable.

index (plural indices) See exponent.

integer A positive or negative whole number (including zero).

integral Pertaining to an integer. For example the statement that m can take integral values from -2 to $+2$ means that the possible values of m are -2,

−1, 0, 1 and 2.

intercept The value on one axis of a graph at which a plotted straight line crosses that axis, provided that axis does pass through the zero point on the other axis. If the plotted line has an equation of form $y = mx + c$, the intercept on the y axis is equal to c.

interpolation The process of reading between data points plotted on a graph, in order to find the value of one or both of the plotted quantities at intermediate positions.

interval level A level of measurement in which the *actual* values of measurements or counts are known and used in statistical analysis (e.g. dry mass in grams, number of flowers per plant). See also categorical level, ordinal level.

inverse cosine x is the inverse cosine (arccosine) of y if x is the angle whose cosine is y. i.e. $x = \cos^{-1} y$ ($x = \arccos y$) if $y = \cos x$.

inverse sine x is the inverse sine (arcsine) of y if x is the angle whose sine is y. i.e. $x = \sin^{-1} y$ ($x = \arcsin y$) if $y = sin x$.

inverse tangent x is the inverse tangent (arctangent) of y if x is the angle whose tangent is y, i.e. $x = \tan^{-1} y$ ($x = \arctan y$) if $y = \tan x$.

inverse trigonometric function If y is a trigonometric ratio of the angle x, then x is the inverse trigonometric function of y. For example, if $y = \sin x$, the inverse trigonometric function is $x = \sin^{-1} y$ (or $\arcsin y$) where $\sin^{-1} y$ ($\arcsin y$) is the angle whose sine is y.

inversely proportional (quantities) Two quantities x and y are said to be inversely proportional to each other if an increase in x by a certain factor automatically results in a decrease in y by the same factor (e.g. if the value of x doubles, then the value of y halves). Inverse proportionality between x and y is indicated by writing $y \propto \frac{1}{x}$. A graph in which y is plotted against x will be a hyperbola. See also: directly proportional.

irrational number A number that cannot be obtained by dividing one integer by another, e.g. π, $\sqrt{2}$ and e. See also rational number.

latitude Part of the specification of the position of a point on the Earth's surface: the distance north or south of the Equator measured in degrees. A line of latitude is an imaginary circle on the surface of the Earth.

level of measurement The three levels of measurement that data may be known or analysed at are categorical level, interval level or ordinal level.

linear scale A scale on which the steps between adjacent divisions correspond to the addition or subtraction of a fixed quantity.

logarithm The logarithm of a number to a given base is the power to which the base must be raised in order to produce the number.

logarithm to base 10 The logarithm to base 10 (or 'common logarithm', $\log_{10}$) of p is the power to which 10 must be raised in order to equal p. i.e. if $p = 10^n$, then $\log_{10} p = n$.

logarithm to base e The logarithm to base e (or 'natural logarithm') of p is the power to which e must be raised in order to equal p, i.e. if $p = e^q$, then $\ln p = q$.

logarithmic scale Scale on which the steps between adjacent divisions correspond to multiplication or division by a fixed amount, usually a power of ten.

log-linear graph A graph of the logarithm of one quantity against the actual value of another quantity. For an exponential function of the type $y = Ce^{kx}$, graphs of $\log_{10} y$ against x and of $\ln y$ against x will both be straight lines.

log-log graph A graph of the logarithm of one quantity against the logarithm of another quantity. For a function of the type $y = ax^b$ (e.g. $y = 2x^3$) graphs of $\log_{10} y$ against $\log_{10} x$ and of $\ln y$ against $\ln x$ will both be straight lines.

longitude Part of the specification of the position of a point on the Earth's surface. A line of longitude is an imaginary semicircle that runs from one pole to the other. The line of zero longitude passes through Greenwich in London. Other lines of longitude are specified by the angle east or west of the line of zero longitude.

lowest common denominator The smallest common denominator of two or more fractions.

magnitude The size of a quantity, also referred to as the 'modulus'. Vector quantities have both magnitude and direction; scalar quantities have only magnitude.

matched samples When data are collected from two samples such that each item of data from one sample can be uniquely matched with just one item of data from the other sample (e.g. blood glucose levels measured in individuals before and after they have taken medication), the samples are described as matched. See also unmatched samples.

mean Term commonly used as an abbreviation for arithmetic mean.

median The middle value in a series when the values are arranged in either increasing or decreasing order. If the series contains an odd number of items, the median is the value of the middle item; if it contains an even number of items, the median is the arithmetic mean of the values of the middle two items.

minute (of arc) A 60th part of an degree (of arc).

mixed number A number consisting of a non-zero integer and a fraction, e.g. $3\frac{1}{2}$. Any improper fraction may also be written as a mixed number: for example $\frac{8}{3} = 2\frac{2}{3}$.

mode The most frequently occurring value in a set of data.

modulus See magnitude.

multiplication rule for probabilities A rule stating that if a number of outcomes occur independently of one another, the probability of them all happening together is found by multiplying the individual probabilities.

natural logarithm See logarithm to base e.

normal distribution Distribution of measurements or characteristics which lie on a bell-shaped curve that is symmetric about its peak, with the peak corresponding to the mean value. Repeated independent measurements of the same quantity approximate to a normal distribution, as do quantitative characters in natural populations (e.g. height in human beings).

null hypothesis A 'no difference' hypothesis proposed at the start of a statistical hypothesis test (e.g. that the means of two populations are identical, $\mu_1 = \mu_2$). Compare with alternative hypothesis.

numerator The number or term on the top of a fraction. For example, in the fraction $\frac{3}{4}$, the numerator is 3; in the fraction $\frac{a+b}{c}$, the numerator is $a + b$. See also denominator.

opposite (trigonometry) The side opposite to a particular angle in a right-angled triangle.

order of magnitude The approximate value of a quantity, expressed as the nearest power of ten. If the value of the quantity is expressed in scientific notation as $a \times 10^n$, then the order of magnitude of the quantity is 10^n if $a < 5$ and 10^{n+1} if $a > 5$. The phrase is also used to compare the sizes of quantities, as in 'a metre is three orders of magnitude longer than a millimetre' or 'a picogram is twelve orders of magnitude smaller than a gram'.

ordinal level A level of measurement in which the data can be logically ranked but in which the *actual* values of the measurements or counts are either not known or not used in statistical analysis (e.g. tallest to shortest, heaviest to lightest). See also categorical level, interval level.

origin (of a graph) The point on a graph at which the quantities plotted on the horizontal axis and the vertical axis are both zero.

parabola A curve that may be described by an equation of the form $y = ax^2 + bx + c$, where x and y are variables, a is a non-zero constant, and b and c are constants that may take any value.

percentage A way of expressing a fraction with a denominator of 100. For example, 12 per cent (also written 12%) is equivalent to twelve parts per hundred or $\frac{12}{100}$.

places of decimals In decimal notation, the number of digits after the decimal point (including zeroes). Thus 21.327 and 3.000 are both given to three places of decimals.

population Statistical term used to describe the complete set of things or events being studied.

power See exponent.

powers of ten notation A method of representing a number as a larger or smaller number multiplied by ten raised to the appropriate power. For example, 2576 can be written in powers of ten notation as 25.76×10^2 or 2.576×10^3, or 0.02576×10^5 or 257600×10^{-2}. See also scientific notation.

precise Description of a set of measurements for which the random uncertainty is small. Compare with accurate.

probability If a process is repeated a very large number if times, then the probability of a particular outcome may be defined in terms of results obtained as the fraction of results corresponding to that particular outcome. If the process has n equally likely outcomes and q of those outcomes correspond to a particular event, then the probability of that event is defined as q/n. There are, for example, 6 equally likely outcomes for the process of rolling a fair die. Only one of those outcomes corresponds to the event 'throwing a six', so the probability of throwing a six is $\frac{1}{6}$. Five of the outcomes correspond to the event 'not throwing a six', so the probability of not throwing a six is $\frac{5}{6}$.

product The result of a multiplication operation. For example, the product of 3 and 5 is 15.

proportional See directly proportional, inversely proportional.

Pythagoras' Theorem The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.

quadratic equation An algebraic equation for x of the form $ax^2 + bx + c = 0$, where $a \neq 0$ and b and c can take any value. For example, $2x^2 - x + 3 = 0$ is a quadratic equation.

quadratic equation formula The solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

radian The angle subtended at the centre of a circle by an arc equal in length to the radius. In general, the angle θ subtended by an arc length s in a circle of radius r is given by θ (in radians) $= \frac{s}{r}$.

random uncertainty Measured values of one quantity that are scattered over a limited range about a mean value are said to be subject to random uncertainty. The larger the random uncertainty associated with the measurements, the larger will be the scatter. See also precise and systematic uncertainty.

ratio The relationship between the sizes of two comparable quantities. For example, if a group of 11 people is made up of 8 women and 3 men, the ratio of women to men is said as 8 to 3 and written as 8 : 3. Ratios may be fairly easily converted into fractions. In this particular example $\frac{8}{8+3} = \frac{8}{11}$ of the group are women and $\frac{3}{11}$ are men.

rational number Any number that can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$, e.g. $7 = \frac{7}{1}$; $-6 = \frac{-6}{1}$; $-\frac{1}{3}$; $3.125 = \frac{25}{8}$. Every terminating or recurring decimal is a rational number. See also: irrational number.

real number A number that can be placed on the number line. The set of real numbers is made up of all the rational and irrational numbers.

reciprocal A term that is related to another as $\frac{2}{3}$ is related to $\frac{3}{2}$. The reciprocal of $\frac{y}{x}$ is $\frac{x}{y}$, and vice versa, for any non-zero values of x and y. The reciprocal of N^m is N^{-m} and vice versa.

recurring decimal A number in which the pattern of digits after the decimal point repeats itself indefinitely. Every recurring decimal is a rational number and can therefore be written as a fraction, e.g. $0.3333\ldots = \frac{1}{3}$; $0.123\,123\,123\ldots = \frac{41}{333}$; $0.2345\,2345\,2345\ldots = \frac{2345}{9999}$.

right angle The angle between two directions that are perpendicular (i.e. at 90°) to each other.

right-angled triangle A triangle where the angle between two of the sides is a right angle.

rounding error An error introduced into a calculation by working to too few significant figures. To avoid rounding errors you should work to at least one more significant figure than is required in the final answer, and just round at the end of the whole calculation.

sample Statistical term used to describe an unbiased sub-set of a population.

sample standard deviation See estimated standard deviation of a population.

scalar A quantity with magnitude but no direction. Compare with vector.

scientific notation Method of writing numbers, according to which any rational number can be written in the form $a \times 10^n$ where a is either an integer or a number written in decimal notation, $1 \leq a < 10$, and n is an integer. Thus 5 870 000 may be written in scientific notation as 5.87×10^6, and 0.003 261 may be written in scientific notation as 3.261×10^{-3}. The terms 'standard form' and 'standard index form' are equivalent to the term scientific notation.

second (of arc) See arcsec.

second derivative A derivative of a derivative, for example the derivative of $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ with respect to x. A second derivative is usually written as or $\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ or $f''(x)$.

SI units An internationally agreed system of units. In this system, there are seven base units (which include the metre, kilogram and the second) and an unlimited number of derived units obtained by combining the base units in various ways. The system recognizes a number of standard abbreviations (of which SI, standing for Système International, is one). The system also uses certain standard multiples and submultiples, represented by standard prefixes. See also centi and deci.

significance level The probability that the value of a test statistic could be as extreme (usually as large, but in some statistical tests as small) as the value obtained in a statistical hypothesis test if the null hypothesis were true.

significant figures The number of digits, excluding leading zeroes, quoted for the value of a quantity, and defined as the number of digits known with certainty plus one uncertain digit. Thus if a measured temperature is given as 23.7°C (i.e. quoted to three significant figures) this implies that the first two digits are certain, but there is some uncertainty in the final digit, so the real temperature might be 23.6°C or 23.8°C. The larger the number of significant figures quoted for a value, the smaller is the uncertainty in that value. Leading zeroes in decimal numbers do not count as significant figures (e.g. 0.002 45 is expressed to three significant figures). Numbers equal to or greater than 100 can be unambiguously expressed to two significant figures only by the use of scientific notation (e.g. 450 can only be unambiguously expressed to two significant figures by writing it in the form 4.5×10^2). Similarly, scientific notation must be used to express numbers equal to or greater than 1000 unambiguously to 3 significant figures.

similar Two triangles (or other objects) are described as being similar if they have the same shape but different size.

simplify To write an equation or expression in its simplest form.

simultaneous equations Two or more equations which must hold true simultaneously.

sine The sine of an angle q in a right-angled triangle is defined by

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

where 'opposite' is the length of the side opposite θ and 'hypotenuse' is the length of the hypotenuse.

sketch graph A graph drawn to illustrate the nature of the relationship between quantities, but not involving accurate plotting. On a sketch graph the origin is usually indicated, but the axes are not scaled.

skewed Description of distributions that are not symmetric about their mean value.

small angle approximation For small angles (less than about 0.1 radian) $\cos\theta \approx 1$, and if the angle is stated in radians, $\sin\theta \approx \theta$, $\tan\theta \approx \theta$.

solution The answer, especially numerical value or values which satisfy an algebraic equation.

solve To find an answer, usually to find the numerical values which satisfy an algebraic equation.

Spearman rank correlation coefficient (r_s) A test statistic calculated in a statistical hypothesis test used to determine whether or not there is a statistically significant correlation between two ordinal level variables.

square root The number or expression that multiplied by itself gives N is called the square root of N. The positive square root of N can be written as either $\sqrt{N}$ or $N^{\frac{1}{2}}$.

standard deviation A quantitative measure of the spread of a set of measurements. For n repeated measurements of a quantity, with arithmetic mean $\overline{x}$, the standard deviation s_n is given by

$$s_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

The symbol σ_n is also widely used (especially on calculators) as an equivalent to s_n. See also: sample standard deviation, estimated standard deviation of a population.

standard form See scientific notation.

standard index form See scientific notation.

statistically significant In science, the result of a statistical hypothesis test is conventionally regarded as statistically significant if the probability of the value of the test statistic being as large (or, in some statistical tests, as small) as the one obtained is less than 0.05.

subject The term written by itself, usually to the left of the equals sign in a mathematical equation.

subtend A straight line rotating about a certain point is said to subtend the angle it passes through.

sum The result of an addition operation. For example, the sum of 3 and 2 is 5. A summation sign may be used as shorthand for more complicated addition operations, e.g.

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n.$$

systematic uncertainty Measured values of one quantity that are consistently too large or too small because of bias in the measuring instrument or the measurement technique are said to be subject to systematic uncertainty. See also accurate, random uncertainty.

t-test One of a number of statistical tests of a hypothesis used to determine whether there is a statistically significant difference between the estimated population means calculated from two samples. Different versions of the test are available for matched samples and unmatched samples.

tangent (to a curved graph) The tangent to a curve at a given point P is the straight line that just touches the curve at P and has the same gradient as the curve at the point P.

tangent (trigonometry) The tangent of an angle θ in a right-angled triangle is defined by

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

where 'opposite' is the length of the side opposite and 'adjacent' is the length of the side adjacent to θ.

term A single variable (such as v_x or u_x in the equation $v_x = u_x + a_x t$) or a combination of variables, such as $a_x t$.

test of association A statistical hypothesis test used to determine whether there is a statistically significant association between two categorical level variables (e.g. χ^2 test) or a statistically significant correlation between two variables at ordinal level (e.g. Spearman rank correlation (r_s)) or at interval level (other correlation coefficients (r)).

test of difference A statistical hypothesis test used to test whether there is a statistically significant difference between, for example, the estimated population means (e.g. t-tests) or estimated population medians (other tests) calculated from two samples.

test statistic In most statistical tests of a hypothesis, the value of a test statistic is calculated using an equation. The value of the test statistic is then compared with a table of critical values in order to determine whether the null hypothesis ought to be accepted or rejected at a particular significance level.

trigonometric ratios The ratios of the sides of a right-angled triangle, including tangent, sine, cosine.

trigonometry The branch of mathematics which deals with the relations between the sides and angles of triangles, usually right-angled triangles.

true mean The arithmetic mean of some quantity for a whole population, usually denoted by the symbol μ. For a large population, the true mean is generally unknowable and the best estimate that can be made of it is the mean of the quantity for an unbiased sample drawn from the population.

unmatched samples When data are collected from two samples such that there is no logical connection between any particular item of data from one sample and any particular item of data from the other sample (e.g. the heights of plants randomly assigned to either an experimental or a control group), the samples are described as unmatched. See also matched samples.

variable A quantity that can take a number of values.

vector A physical quantity that has a definite magnitude and points in a definite direction.

word equation An equation in which the quantities under consideration are described in words.

Hidden material

This ‘chapter’ contains material which you won’t normally read through in sequence, but will access it through the links from the main text.

Question 1.1 (a)

$$(-3) \times 4 = -12$$

Question 1.1 (b)

$(-10) - (-5) = -5$

Question 1.1 (c)

$6 \div (-2) = -3$

Question 1.1 (d)

$$(-12) \div (-6) = 2$$

Question 1.2

The lowest temperature in the oceans, which corresponds to the freezing point, is 31.9 Celsius degrees colder than the highest recorded temperature, which is 30.0 °C.

$$\begin{aligned}\text{Therefore, freezing point of seawater} &= 30.0\ ^\circ\text{C} - 31.9\ ^\circ\text{C}\\ &= -1.9\ ^\circ\text{C}\end{aligned}$$

Question 1.3 (a)

$$117 - (-38) + (-286) = -131$$

Question 1.3 (b)

$$(-1624) \div (-29) = 56$$

Question 1.3 (c)

$(-123) \times (-24) = 2952$

Question 1.4 (a)

The lowest common denominator is 6, so

$$\frac{2}{3} - \frac{1}{6} = \frac{2 \times 2}{3 \times 2} - \frac{1}{6} = \frac{4}{6} - \frac{1}{6} = \frac{3}{6}$$

Dividing top and bottom by 3 gives

$$\frac{3}{6} = \frac{1}{2}$$

Alternatively,

$$\frac{2}{3} - \frac{1}{6} = \frac{2 \times 6}{3 \times 6} - \frac{1 \times 3}{6 \times 3} = \frac{12}{18} - \frac{3}{18} = \frac{9}{18}$$

Dividing top and bottom by 9 gives

$$\frac{9}{18} = \frac{1}{2}$$

as before.

Question 1.4 (b)

The lowest common denominator is 30, so

$$\begin{aligned}\frac{1}{3}+\frac{1}{2}-\frac{2}{5} &= \frac{1\times 10}{30}+\frac{1\times 15}{30}-\frac{2\times 6}{30}\\ &= \frac{10}{30}+\frac{15}{30}-\frac{12}{30}\\ &= \frac{13}{30}\end{aligned}$$

Question 1.4 (c)

In this case, the lowest common denominator isn't immediately obvious, but a common denominator will certainly be given by the product of 3 and 28, so

$$\begin{aligned}\frac{5}{28} - \frac{1}{3} &= \frac{5 \times 3}{28 \times 3} - \frac{1 \times 28}{3 \times 28} \\ &= \frac{15}{84} - \frac{28}{84} \\ &= -\frac{13}{84}\end{aligned}$$

Question 1.5 (a)

The original fraction, $\frac{4}{16} = \frac{1}{4} = 0.25$.

You may have chosen any number for your calculations. In this answer the number 2 is used, but the principles hold good whatever choice of (non-zero) number is made.

Suppose we were to add 2 to the numerator and to the denominator

$$\frac{4+2}{16+2} = \frac{6}{18} = 0.333 \text{ to three places of decimals}$$

This is not the same as the original fraction. (There is just one special case in which this kind of operation would not change the value of the fraction and that is adding 0 to top and bottom, which obviously leaves the fraction unchanged.)

Question 1.5 (b)

Suppose we were to subtract 2 from the numerator and from the denominator

$$\frac{4-2}{16-2} = \frac{2}{14} = 0.143 \text{ to three places of decimals}$$

This is not the same as the original fraction. (Again, subtracting 0 from top and bottom is the only case in which this operation leaves the fraction unchanged.)

Question 1.5 (c)

If we square the numerator and the denominator

$$\frac{4 \times 4}{16 \times 16} = \frac{16}{256} = 0.0625$$

This is not the same as the original fraction.

Question 1.5 (d)

If we take the square root of the numerator and of the denominator

$$\frac{\sqrt{4}}{\sqrt{16}} = \frac{2}{4} = 0.5$$

This is not the same as the original fraction.

Incidentally, checking a general rule by trying out a specific numerical example is a helpful technique, which will be useful for algebra in Chapter 4.

Question 1.6 (a)

$$\frac{2}{7} \times 3 = \frac{2 \times 3}{7} = \frac{6}{7}$$

Question 1.6 (b)

$$\frac{5}{9} \div 7 = \frac{5}{9} \times \frac{1}{7} = \frac{5 \times 1}{9 \times 7} = \frac{5}{63}$$

Question 1.6 (c)

$$\frac{1/6}{1/3} = \frac{1}{6} \div \frac{1}{3} = \frac{1}{6} \times \frac{3}{1} = \frac{3}{6} = \frac{1}{2}$$

Question 1.6 (d)

$$\frac{3}{4} \times \frac{7}{8} \times \frac{2}{7} = \frac{3 \times 7 \times 2}{4 \times 8 \times 7} = \frac{42}{224}$$

Dividing top and bottom by 2, and then by 7

$$\frac{42}{224} = \frac{21}{112} = \frac{3}{16}$$

Alternatively, the original could have been simplified in the same way before carrying out any multiplication:

$$\frac{3}{\cancel{4}_2} \times \frac{\cancel{7}^1}{8} \times \frac{\cancel{2}^1}{\cancel{7}_1} = \frac{3}{16}$$

Question 1.7 (a)

$$2^{-2} = \frac{1}{2^2} = \frac{1}{2 \times 2} = \frac{1}{4}$$

You might have gone one step further and expressed this in decimal notation as 0.25.

Question 1.7 (b)

$$\frac{1}{3^{-3}} = 3^3 = 3 \times 3 \times 3 = 27$$

Question 1.7 (c)

$$\frac{1}{4^0} = \frac{1}{1} = 1$$

Question 1.7 (d)

$$\frac{1}{10^4} = \frac{1}{10\,000} = 0.000\,1$$

Question 1.8 (a)

$2^9 = 512$

Question 1.8 (b)

$$3^{-3} = \frac{1}{3^3} = 0.037 \text{ to three places of decimals}$$

It doesn't matter if you quoted more digits in your answer than this. There is more explanation in Chapter 2 about how and when to round off the values given on your calculator display.

Question 1.8 (c)

$$\frac{1}{4^2} = 4^{-2} = 0.0625$$

Question 1.9 (a)

$2^{30} \times 2^2 = 2^{(30+2)} = 2^{32}$

Question 1.9 (b)

$$3^{25} \times 3^{-9} = 3^{(25+(-9))} = 3^{16}$$

Question 1.9 (c)

$10^2/10^3 = 10^2 \div 10^3 = 10^{(2-3)} = 10^{-1}$ (or $1/10$)

Question 1.9 (d)

$$10^2/10^{-3} = 10^2 \div 10^{-3} = 10^{(2-(-3))} = 10^5$$

or alternatively

$$10^2/10^{-3} = 10^2 \times \frac{1}{10^{-3}} = 10^2 \times 10^3 = 10^5$$

Question 1.9 (e)

$10^{-4} \div 10^{2} = 10^{(-4-2)} = 10^{-6}$

Question 1.9 (f)

$$\frac{10^5 \times 10^{-2}}{10^3} = 10^{(5+(-2)-3)} = 10^0 \text{ (or 1)}$$

Question 1.10 (a)

$$\left(4^{16}\right)^2 = 4^{16\times 2} = 4^{32}$$

Question 1.10 (b)

$$\left(5^{-3}\right)^2 = 5^{(-3)\times 2} = 5^{-6}$$

This could also be written as $\dfrac{1}{5^6}$.

Question 1.10 (c)

$$\left(10^{25}\right)^{-1} = 10^{25\times(-1)} = 10^{-25}$$

This could also be written as $\dfrac{1}{10^{25}}$.

Question 1.10 (d)

$$\left(\frac{1}{3^3}\right)^6 = \frac{1^6}{(3^3)^6} = \frac{1}{3^{3\times 6}} = \frac{1}{3^{18}}$$

or alternatively

$$\left(\frac{1}{3^3}\right)^6 = \left(3^{-3}\right)^6 = 3^{-3\times 6} = 3^{-18} = \frac{1}{3^{18}}$$

Question 1.11 (a)

From Equation 1.3

$$\left(2^4\right)^{\frac{1}{2}} = 2^{\left(4\times\frac{1}{2}\right)} = 2^2 = 4$$

Question 1.11 (b)

From Equation 1.3

$$\sqrt{10^4} = \left(10^4\right)^{\frac{1}{2}} = 10^{4\times\frac{1}{2}} = 10^2 = 100$$

Question 1.11 (c)

From Equation 1.3

$$100^{\frac{3}{2}} = \left(100^{\frac{1}{2}}\right)^3 = 10^3 = 1000$$

Alternatively

$$100^{\frac{3}{2}} = \left(100^3\right)^{\frac{1}{2}} = \left(10^6\right)^{\frac{1}{2}} = 10^{6/2} = 10^3 = 1000$$

Question 1.11 (d)

$$125^{-1/3} = \frac{1}{125^{1/3}} = \frac{1}{5} = 0.2$$

Since the cube root of 125 is 5.

Question 1.12 (a)

Multiplication takes precedence over subtraction, so

$$\begin{aligned} 35 - 5 \times 2 &= 35 - (5 \times 2) \\ &= 35 - 10 \\ &= 25 \end{aligned}$$

Question 1.12 (b)

Here the brackets take precedence, so

$$\begin{aligned}(35 - 5) \times 2 &= 30 \times 2\\ &= 60\end{aligned}$$

Question 1.12 (c)

Again, the brackets take precedence over the (implied) multiplication, so

$$\begin{aligned} 5(2-3) &= 5 \times (-1) \\ &= -5 \end{aligned}$$

Question 1.12 (d)

Here the exponent takes precedence:

$$\begin{aligned} 3 \times 2^2 &= 3 \times 4 \\ &= 12 \end{aligned}$$

Question 1.12 (e)

The exponent takes precedence again:

$$\begin{aligned} 2^3 + 3 &= 8 + 3 \\ &= 11 \end{aligned}$$

Question 1.12 (f)

Here both brackets take precedence over the (implied) multiplication:

$$\begin{aligned}(2+6)(1+2) &= 8 \times 3 \\ &= 24\end{aligned}$$

Question 2.1 (a)

$$
\begin{aligned}
5.4 \times 10^4 &= 5.4 \times 10\,000 \\
&= 54\,000
\end{aligned}
$$

Question 2.1 (b)

$$
\begin{aligned}
2.1 \times 10^{-2} &= 2.1 \times \frac{1}{100} \\
&= \frac{2.1}{100} \\
&= 0.021
\end{aligned}
$$

Question 2.1 (c)

$$
\begin{aligned}
0.6 \times 10^{-1} &= 0.6 \times \frac{1}{10} \\
&= \frac{0.6}{10} \\
&= 0.06
\end{aligned}
$$

Question 2.2 (a)

$$\begin{aligned}215 &= 2.15 \times 100\\ &= 2.15 \times 10^2\end{aligned}$$

Question 2.2 (b)

$$
\begin{aligned}
46.7 &= 4.67 \times 10 \\
&= 4.67 \times 10^1
\end{aligned}
$$

Question 2.2 (c)

$$
\begin{aligned}
152 \times 10^3 &= 1.52 \times 100 \times 10^3 \\
&= 1.52 \times 10^2 \times 10^3 \\
&= 1.52 \times 10^{(2+3)} \\
&= 1.52 \times 10^5
\end{aligned}
$$

Question 2.2 (d)

$$
\begin{aligned}
0.000\,0876 &= \frac{8.76}{100\,000} \\
&= \frac{8.76}{10^5} \\
&= 8.76 \times 10^{-5}
\end{aligned}
$$

Question 2.3 (a)

A kilometre is 10^3 times bigger than a metre, so

$$\begin{aligned} 3476\ \text{km} &= 3.476 \times 10^3\ \text{km} \\ &= 3.476 \times 10^3 \times 10^3\ \text{m} \\ &= 3.476 \times 10^6\ \text{m} \end{aligned}$$

Question 2.3 (b)

A micrometre is 10^3 times bigger than a nanometre, so

$$8.0\ \mu\text{m} = 8.0 \times 10^3\ \text{nm}$$

Question 2.3 (c)

A second is 10^3 times bigger than a millisecond, so

$$0.8 \text{ s} = 0.8 \times 10^3 \text{ ms}$$

To express this in scientific notation, we need to multiply and divide the right-hand side by 10:

$$\begin{aligned} 0.8 \times 10^3 \text{ ms} &= (0.8 \times 10) \times \frac{10^3}{10} \text{ ms} \\ &= 8 \times \left(10^3 \times 10^{-1}\right) \text{ ms} \\ &= 8 \times 10^{(3-1)} \text{ ms} \\ &= 8 \times 10^2 \text{ ms} \end{aligned}$$

Question 2.4 (a)

One million = 10^6, so the distance is

$$\begin{aligned} 5900 \times 10^6\ \text{km} &= 5.9 \times 10^9\ \text{km} \\ &\sim 10^{10}\ \text{km (or } 10^{13}\ \text{m)} \end{aligned}$$

Question 2.4 (b)

The diameter of a spherical object is given by twice its radius. So for the Sun,

$$
\begin{aligned}
\text{diameter} &= 2 \times 6.97 \times 10^7 \text{ m} \\
&= 13.94 \times 10^7 \text{ m} \\
&= 1.394 \times 10^8 \text{ m} \\
&\sim 10^8 \text{ m}
\end{aligned}
$$

Question 2.4 (c)

$$
\begin{aligned}
2\pi &= 2 \times 3.14 \text{ (to two places of decimals)} \\
&= 6.28
\end{aligned}
$$

This is greater than 5, so can be rounded up to the next power of ten to give the order of magnitude, i.e. $2\pi \sim 10$ (or 10^1).

Question 2.4 (d)

$$\begin{aligned} 7.31 \times 10^{-26}\ \text{kg} &\sim 10 \times 10^{-26}\ \text{kg} \\ &\sim 10^{(-26+1)}\ \text{kg} \\ &\sim 10^{-25}\ \text{kg} \end{aligned}$$

Question 2.5 (a)

(i) 10^0 m = 1 m and 10^{-2} m = 0.01 m, so the difference between them is $(1 - 0.01)$ m = 0.99 m.

(ii) 10^2 m = 100 m and 10^0 m = 1 m, so the difference between them is 99 m.

(iii) 10^4 m = 10 000 m and 10^2 m = 100 m, so the difference between them is 9900 m.

It is quite clear that as one goes up the scale the interval between each successive pair of tick marks increases by 100 times.

Question 2.5 (b)

The height of a child is about 10^0 m, i.e. 1 m. The height of Mount Everest is about 10^4 m (actually 8800 m, but it is not possible to read that accurately from the scale on Figure 2.2). So Mount Everest is $\sim 10^4$ times taller than a child.

Question 2.5 (c)

The length of a typical virus is 10^{-8} m and the thickness of a piece of paper is 10^{-4} m, so it would take $\sim 10^{-4}/10^{-8} = 10^{-4-(-8)} = 10^{-4+8} = 10^{4}$ viruses laid end to end to stretch across the thickness of a piece of paper.

Question 2.6

Magnitude 7 on the Richter scale represents four points more than magnitude 3, and each point increase represents a factor 10 increase in maximum ground movement. So a magnitude 7 earthquake corresponds to 10^4 (i.e. 10 000) times more ground movement than a magnitude 3 earthquake.

Question 2.7

Each of the quantities is quoted to four significant figures.

Question 2.8 (a)

The third digit is an 8, so the second digit must be rounded up:

$-38.87\ ^{\circ}\mathrm{C} = -39\ ^{\circ}\mathrm{C}$ to two significant figures

Question 2.8 (b)

There is no way of expressing a number greater than or equal to 100 unambiguously to two significant figures except by the use of scientific notation. The third digit is a 5, so again the second digit must be rounded up.

$$\begin{aligned} -195.8\ ^\circ\mathrm{C} &= -1.958 \times 10^2\ ^\circ\mathrm{C} \\ &= -2.0 \times 10^2\ ^\circ\mathrm{C} \text{ to two significant figures} \end{aligned}$$

{Note that the final zero does count.}

Question 2.8 (c)

Again, this quantity cannot be expressed unambiguously to two significant figures without the use of scientific notation. The third digit is an 8, so the second digit must be rounded up.

$$\begin{aligned} 1083.4\ ^\circ\mathrm{C} &= 1.0834 \times 10^3\ ^\circ\mathrm{C} \\ &= 1.1 \times 10^3\ ^\circ\mathrm{C} \text{ to two significant figures} \end{aligned}$$

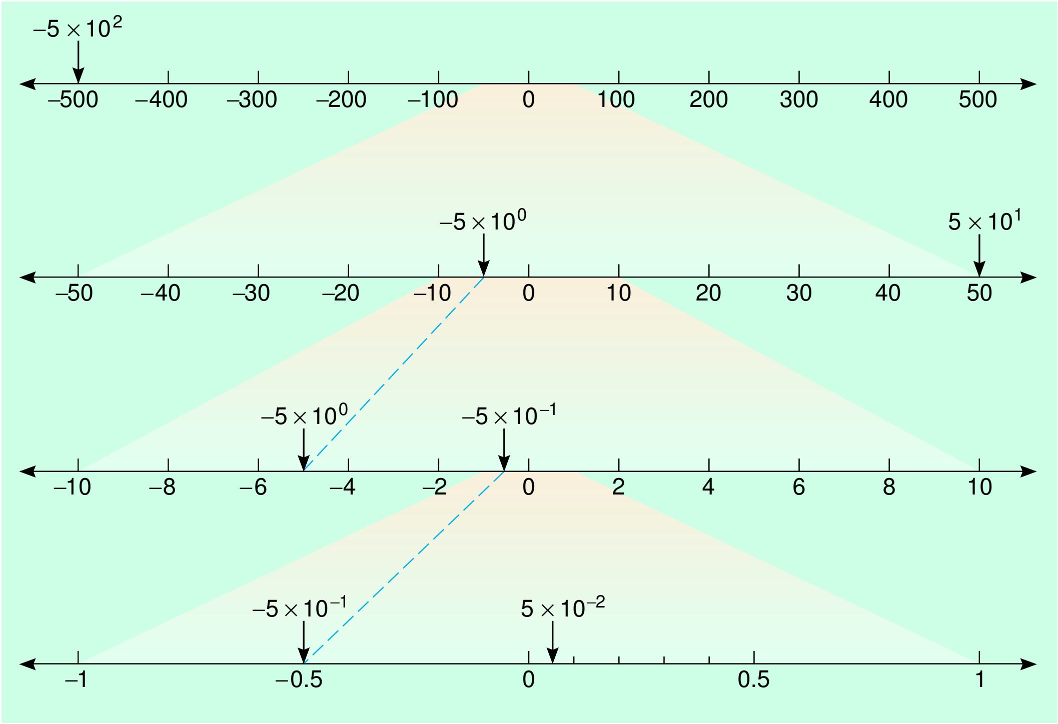


Figure 2.1: Portions of the number line, showing the positions of a few large and small numbers expressed in scientific notation.

Click on Back to return to text

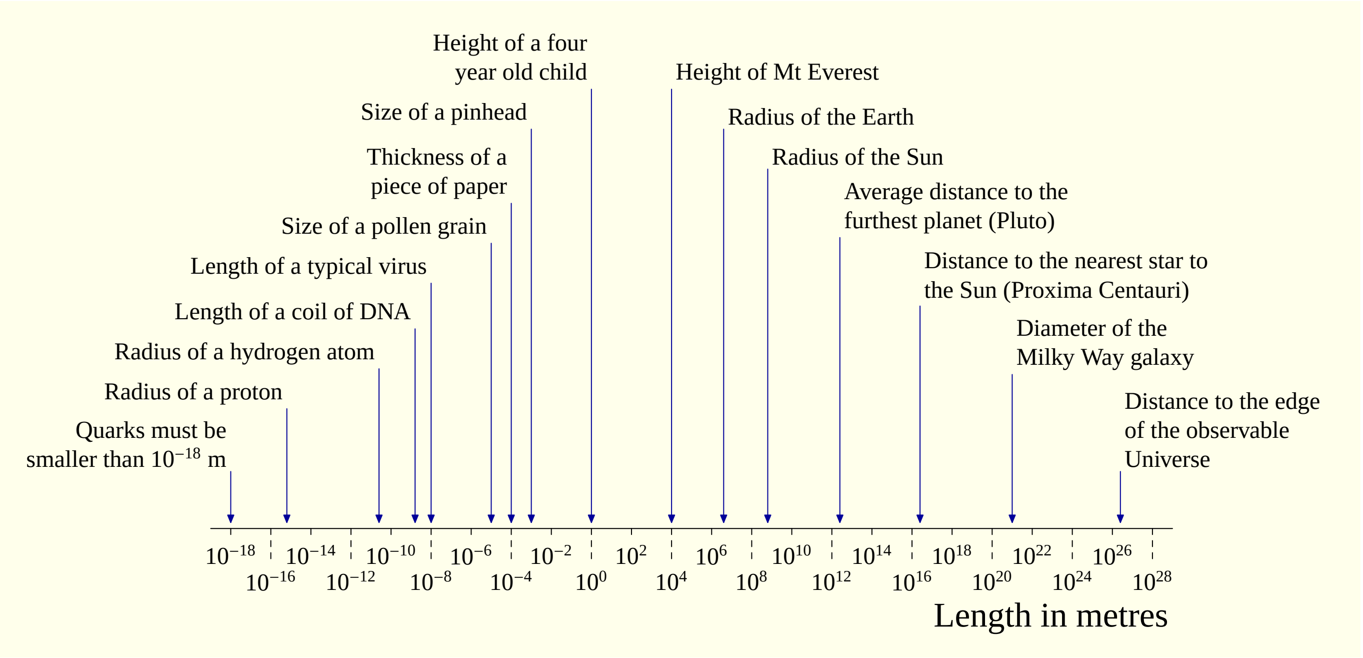


Figure 2.2: The scale of the known Universe.

Click on Back to return to text

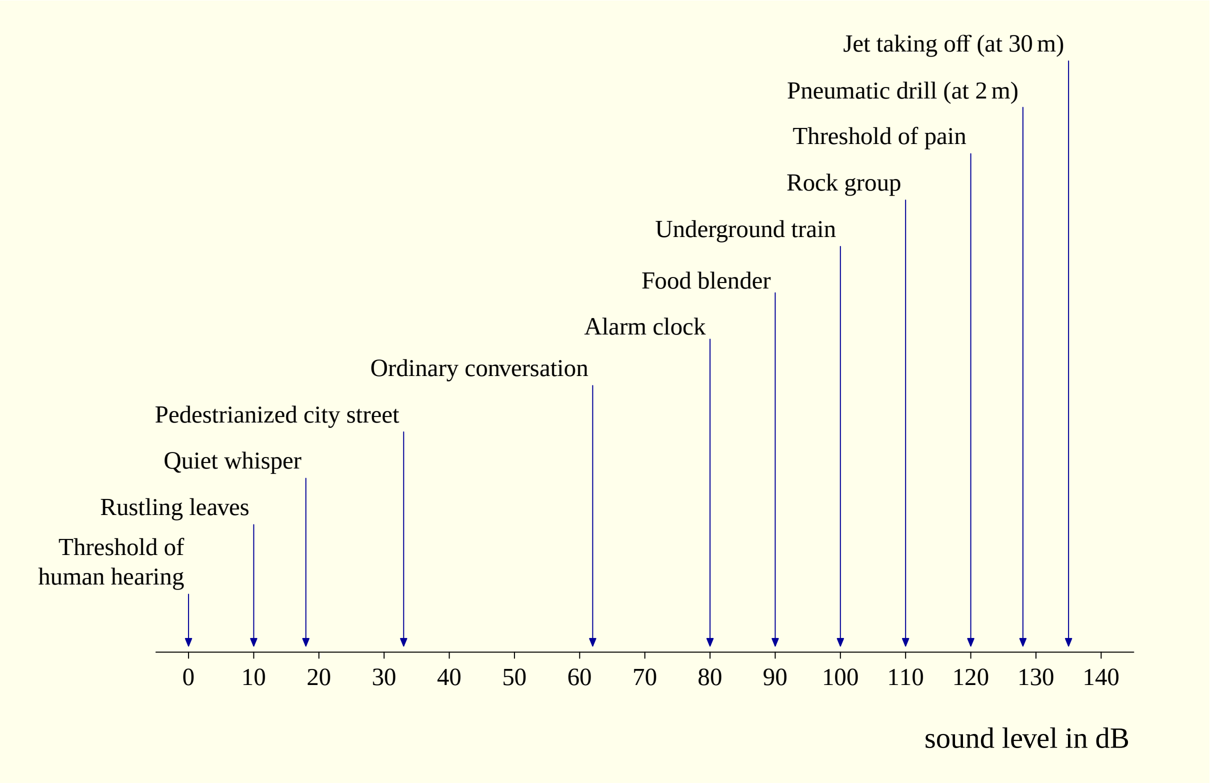


Figure 2.3: Some common sounds on the decibel scale of sound level.

Click on Back to return to text

Question 3.1

$(\text{inch})^2$, cm^2 and square miles all have units of $(\text{length})^2$, so they are all units of area.

s^2 cannot be a unit of area because the unit which has been squared, the second, is a unit of time not of length.

m^{-2} cannot be a unit of area because the metre is raised to the power *minus* 2, not 2.

km^3 cannot be a unit of area because the kilometre is cubed not squared. In fact, it is a unit of volume.

Question 3.2 (a)

$\dfrac{6.732}{1.51} = 4.458 = 4.46$ to three significant figures.

{6.732 is known to four significant figures, and 1.51 is known to three significant figures. The number of significant figures in the answer is the same as in the input value with the fewest significant figures, i.e. three.}

Question 3.2 (b)

$2.0 \times 2.5 = 5.0$ to two significant figures.

{2.0 and 2.5 are both given to two significant figures, so the answer is given to two significant figures too.}

Question 3.2 (c)

Working to three significant figures and rounding to two significant figures at the end of the calculation gives:

$$\left(\frac{4.2}{3.1}\right)^2 = (1.35)^2 = 1.82 = 1.8 \text{ to two significant figures.}$$

{Squaring is repeated multiplication, so it is reasonable to quote the final answer to two significant figures. However, working to two significant figures throughout introduces a sizeable rounding error and gives a final answer of 2.0.}

Question 3.2 (d)

The total mass = 3×1.5 kg = 4.5 kg.

{Note that you have exactly 3 bags of flour, so it would not be correct to round the answer to one significant figure.}

Question 3.3 (a)

$$\begin{aligned}(3.0 \times 10^6) \times (7.0 \times 10^{-2}) &= (3.0 \times 7.0) \times 10^{6+(-2)}\\ &= 21 \times 10^4\\ &= 2.1 \times 10^5\end{aligned}$$

{Note that 21×10^4 is a correct numerical answer to the multiplication, but it is not given in scientific notation.}

Question 3.3 (b)

$$\frac{8 \times 10^4}{4 \times 10^{-1}} = \frac{8}{4} \times 10^{4-(-1)} = 2 \times 10^5$$

Question 3.3 (c)

$$\frac{10^4 \times (4 \times 10^4)}{1 \times 10^{-5}} = 4 \times \frac{10^{4+4}}{10^{-5}} = 4 \times 10^{8-(-5)} = 4 \times 10^{13}$$

Question 3.3 (d)

$$
\begin{aligned}
\left(3.00 \times 10^{8}\right)^{2} &= (3.00)^{2} \times \left(10^{8}\right)^{2} \\
&= 9.00 \times 10^{8\times 2} \\
&= 9.00 \times 10^{16}
\end{aligned}
$$

Question 3.4

$$
\begin{aligned}
\text{Area} &= \left(9.78 \times 10^{-3}\ \text{m}\right)^2 \\
&= \left(9.78 \times 10^{-3}\right)^2 \text{m}^2 \\
&= 9.56 \times 10^{-5}\ \text{m}^2 \text{ to three significant figures.}
\end{aligned}
$$

Question 3.5

To one significant figure,

$$\text{distance to Proxima Centauri} \approx 4 \times 10^{16}\ \text{m}$$
$$\text{distance to the Sun} \approx 2 \times 10^{11}\ \text{m}$$

Thus,

$$\begin{aligned}\frac{\text{distance to Proxima Centauri}}{\text{distance to the Sun}} &\approx \frac{4 \times 10^{16}\ \text{m}}{2 \times 10^{11}\ \text{m}}\\ &\approx \frac{4}{2} \times \frac{10^{16}\ \text{m}}{10^{11}\ \text{m}}\\ &\approx 2 \times 10^{16-11}\\ &\approx 2 \times 10^{5}\end{aligned}$$

Thus Proxima Centauri is approximately 2×10^{5} times further away than the Sun.

Question 3.6 (a)

$1 \text{ m} = 100 \text{ cm}$, so $1 \text{ m}^2 = 100^2 \text{ cm}^2$

Thus $1.04 \text{ m}^2 = 1.04 \times 100^2 \text{ cm}^2 = 1.04 \times 10^4 \text{ cm}^2$

Question 3.6 (b)

$1 \text{ m} = 10^6 \ \mu\text{m}$, so $1 \text{ m}^2 = \left(10^6\right)^2 \ \mu\text{m}^2$

Thus $1.04 \text{ m}^2 = 1.04 \times \left(10^6\right)^2 \ \mu\text{m}^2 = 1.04 \times 10^{12} \ \mu\text{m}^2$

Question 3.6 (c)

$1 \text{ km} = 10^3 \text{ m}$, so $1 \text{ km}^2 = \left(10^3\right)^2 \text{ m}^2$

Thus $1 \text{ m}^2 = \dfrac{1}{(10^3)^2} \text{ km}^2$

and $1.04 \text{ m}^2 = \dfrac{1.04}{(10^3)^2} \text{ km}^2 = 1.04 \times 10^{-6} \text{ km}^2$

Question 3.7 (a)

1 km = 10^3 m, so 1 km^3 = $\left(10^3\right)^3$ m^3 = 10^9 m^3

$$\begin{aligned}\text{Volume of Mars} &= 1.64 \times 10^{11}\ \text{km}^3\\ &= 1.64 \times 10^{11} \times 10^9\ \text{m}^3\\ &= 1.64 \times 10^{20}\ \text{m}^3\end{aligned}$$

Question 3.7 (b)

$1\text{ m} = 10^3\text{ mm, so }1\text{ m}^3 = \left(10^3\right)^3\text{ mm}^3 = 10^9\text{ mm}^3$

$$\text{Thus }1\text{ mm}^3 = \frac{1}{10^9}\text{ m}^3 = 10^{-9}\text{ m}^3$$

$$\begin{aligned}\text{Volume of ball bearing} &= 16\text{ mm}^3\\ &= 16\times10^{-9}\text{ m}^3\\ &= 1.6\times10^{-8}\text{ m}^3\end{aligned}$$

Question 3.8 (a)

$$1 \text{ m} = 100 \text{ cm}$$

So

$$1 \text{ cm} = \frac{1}{100} \text{ m}$$

Thus

$$1 \text{ cm day}^{-1} = \frac{1}{100} \text{ m day}^{-1}$$

and

$$\begin{aligned} 12 \text{ cm day}^{-1} &= \frac{12}{100} \text{ m day}^{-1} \\ &= 0.12 \text{ m day}^{-1} \end{aligned}$$

Question 3.8 (b)

$$1 \text{ day} = 24 \times 60 \times 60 \text{ s} = 8.64 \times 10^4 \text{ s}$$

So

$$1 \text{ cm day}^{-1} = \frac{1}{8.64 \times 10^4} \text{ cm s}^{-1}$$

and

$$\begin{aligned} 12 \text{ cm day}^{-1} &= \frac{12}{8.64 \times 10^4} \text{ cm s}^{-1} \\ &= 1.4 \times 10^{-4} \text{ cm s}^{-1} \end{aligned}$$

Question 3.9 (a)

$$1 \text{ m} = 10^3 \text{ mm, so } 1 \text{ mm} = \frac{1}{10^3} \text{ m} = 10^{-3} \text{ m}$$

$$1 \text{ year} = 365 \times 24 \times 60 \times 60 \text{ s} = 3.154 \times 10^7 \text{ s}$$

To convert from mm year^{-1} m s^{-1} we need to *multiply* by 10^{-3} (to convert the mm to m) and *divide* by 3.154×10^7 (to convert the year^{-1} to s^{-1}).

$$1 \text{ mm year}^{-1} = \frac{10^{-3}}{3.154 \times 10^7} \text{ m s}^{-1}$$

so

$$\begin{aligned} 0.1 \text{ mm year}^{-1} &= 0.1 \times \frac{10^{-3}}{3.154 \times 10^7} \text{ m s}^{-1} \\ &= 3 \times 10^{-12} \text{ m s}^{-1} \text{ to one significant figure} \end{aligned}$$

So the stalactite is growing at about 3×10^{-12} m s^{-1}.

Question 3.9 (b)

$$1\ \text{m} = 100\ \text{cm, so } 1\ \text{cm} = \frac{1}{100}\ \text{m} = 10^{-2}\ \text{m}$$

$$1\ \text{day} = 24 \times 60 \times 60\ \text{s} = 8.64 \times 10^{4}\ \text{s}$$

To convert from cm day^{-1} to m s^{-1} we need to *multiply* by 10^{-2} (to convert the cm to m) and *divide* by 8.64×10^{4} (to convert the day^{-1} to s^{-1}).

$$1\ \text{cm day}^{-1} = \frac{10^{-2}}{8.64 \times 10^{4}}\ \text{m s}^{-1}$$

$$\begin{aligned} 12\ \text{cm day}^{-1} &= 12 \times \frac{10^{-2}}{8.64 \times 10^{4}}\ \text{m s}^{-1} \\ &= 1.4 \times 10^{-6}\ \text{m s}^{-1} \end{aligned}$$

So the glacier is moving at about 1.4×10^{-6} m s^{-1}.

Question 3.9 (c)

$$1 \text{ km} = 10^3 \text{ m}$$

$$1 \text{ Ma} = 10^6 \times 365 \times 24 \times 60 \times 60 \text{ s} = 3.154 \times 10^{13} \text{ s}$$

To convert from km Ma^{-1} to m s^{-1}, we need to *multiply* by 10^3 (to convert the km to m) and *divide* by 3.154×10^{13} (to convert the Ma^{-1} to s^{-1}).

$$1 \text{ km Ma}^{-1} = \frac{10^3}{3.154 \times 10^{13}} \text{ m s}^{-1}$$

$$\begin{aligned} 35 \text{ km Ma}^{-1} &= 35 \times \frac{10^3}{3.154 \times 10^{13}} \text{ m s}^{-1} \\ &= 1.1 \times 10^{-9} \text{ m s}^{-1} \text{ to two significant figures.} \end{aligned}$$

So the plates are moving apart at an average rate of $1.1 \times 10^{-9} \text{ m s}^{-1}$.

Comparing the answers to parts (a), (b) and (c) shows that the tectonic plates are moving apart approximately 300 times faster than the stalactite is growing. The glacier under consideration moves about 1000 times faster still, but remember that there is considerable variation in the speeds at which all of these processes take place.

Question 3.10 (a)

$$1\ \text{l} = 10^3\ \text{ml}$$

To convert from $\mu\text{g}\,\text{l}^{-1}$ to $\mu\text{g}\,\text{ml}^{-1}$ we need to *divide* by 10^3.

$$1\ \mu\text{g}\,\text{l}^{-1} = \frac{1}{10^3}\ \mu\text{g}\,\text{ml}^{-1} = 10^{-3}\ \mu\text{g}\,\text{ml}^{-1}$$

$$\begin{aligned} 10\ \mu\text{g}\,\text{l}^{-1} &= 10 \times 10^{-3}\ \mu\text{g}\,\text{ml}^{-1} \\ &= 1.0 \times 10^{-2}\ \mu\text{g}\,\text{ml}^{-1} \text{ to two significant figures.} \end{aligned}$$

Question 3.10 (b)

Note that 10 $\mu g\, l^{-1} = 10\ \mu g\, dm^{-3}$, since 1 litre is defined to be equal to 1 dm^3 (Section 3.4.2).

$$1\ mg = 10^3\ \mu g$$

so

$$1\ \mu g = \frac{1}{10^3}\ mg = 10^{-3}\ mg$$

To convert from $\mu g\, dm^3$ to $mg\, dm^3$ we need to *multiply* by 10^{-3}.

$$1\ \mu g\, dm^3 = 10^{-3}\ mg\, dm^3$$

$$\begin{aligned} 10\ \mu g\, dm^3 &= 10 \times 10^{-3}\ mg\, dm^3 \\ &= 1.0 \times 10^{-2}\ mg\, dm^3 \text{ to two significant figures.} \end{aligned}$$

So a concentration of 10 $\mu g\, l^{-1}$ is equal to $1.0 \times 10^{-2}\ mg\, dm^3$.

Question 3.10 (c)

Note that 10 $\mu g\, l^{-1} = 10\ \mu g\, dm^{-3}$.

$$1\ g = 10^6\ \mu g$$

$$\text{so } 1\ \mu g = \frac{1}{10^6}\ g = 10^{-6}\ g$$

$$1\ m = 10\ dm$$

$$\text{so } 1\ m^3 = 10^3\ dm^3$$

$$\text{and } 1\ dm^3 = \frac{1}{10^3}\ m^3 = 10^{-3}\ m^3$$

To convert from $\mu g\, dm^{-3}$ to $g\, m^{-3}$ we need to *multiply* by 10^{-6} (to convert the μg to g) and *divide* by 10^{-3} (to convert the dm^{-3} to m^{-3}).

$$1\ \mu g\, dm^{-3} = \frac{10^{-6}}{10^{-3}}\ g\, m^{-3}$$

$$\begin{aligned}
10\ \mu\text{g}\,\text{dm}^{-3} &= 10 \times \frac{10^{-6}}{10^{-3}}\ \text{g}\,\text{m}^{-3} \\
&= 10 \times 10^{-6-(-3)}\ \text{g}\,\text{m}^{-3} \\
&= 10 \times 10^{-3}\ \text{g}\,\text{m}^{-3} \\
&= 1.0 \times 10^{-2}\ \text{g}\,\text{m}^{-3} \text{ to two significant figures.}
\end{aligned}$$

So a concentration of 10 $\mu\text{g}\,\text{l}^{-1}$ is equal to $1.0 \times 10^{-2}\ \text{g}\,\text{m}^{-3}$.

Question 3.11

(i) and (iii) are equivalent. Multiplication is commutative, so $x(y+z) = (y+z)x$

(ii) and (v) are equivalent. Both multiplication and addition are commutative, so $xy + z = z + yx$

Note that (i) is not equivalent to (ii) since, in (i), the whole of $(y + z)$, not just y, is multiplied by x.

Substituting $x = 3$, $y = 4$ and $z = 5$ gives

(i) $a = x(y+z) = 3 \times (4+5) = 27$

(ii) $a = xy + z = (3 \times 4) + 5 = 17$

(iii) $a = (y+z)x = (4+5) \times 3 = 27$

(iv) $a = x + yz = 3 + (4 \times 5) = 23$

(v) $a = z + yx = 5 + (4 \times 3) = 17$

Question 3.12

The equivalent equations are (i) and (iii), since

$$a\frac{bc^2}{d} = \frac{abc^2}{d} = \frac{bac^2}{d}$$

Note that only the c is squared, so (ii) $m = a\dfrac{b^2c^2}{d}$ and (v) $m = \dfrac{b^2a^2c^2}{d}$ are different.

Only the numerator of the fraction is multiplied by a, so (iv) $m = \dfrac{abc^2}{ad}$ is different too.

Question 3.13

$$NPP = 1.06 \times 10^8 \text{ kJ}$$
$$R = 3.23 \times 10^7 \text{ kJ}$$

From Equation 3.8,

$$\begin{aligned} GPP &= NPP + R \\ &= 1.06 \times 10^8 \text{ kJ} + 3.23 \times 10^7 \text{ kJ} \\ &= 1.38 \times 10^8 \text{ kJ to three significant figures.} \end{aligned}$$

Question 3.14

$\lambda = 621$ nm, $f = 4.83 \times 10^{14}$ Hz

Converting to SI base units gives

$$\lambda = 621 \times 10^{-9}\ \text{m} = 6.21 \times 10^{-7}\ \text{m}$$
$$f = 4.83 \times 10^{14}\ \text{Hz} = 4.83 \times 10^{14}\ \text{s}^{-1}$$

From Equation 3.13,

$$\begin{aligned} v &= f\lambda \\ &= 4.83 \times 10^{14}\ \text{s}^{-1} \times 6.21 \times 10^{-7}\ \text{m} \\ &= 3.00 \times 10^{8}\ \text{m}\,\text{s}^{-1} \text{ to three significant figures.} \end{aligned}$$

{Note that this is the speed of light in a vacuum. Light of this frequency and wavelength is in the red part of the visible spectrum.}

Question 3.15 (a)

From Equation 3.5

$$V = \frac{4}{3}\pi r^3$$

$$r = 6.38 \times 10^3\ \text{km} = 6.38 \times 10^3 \times 10^3\ \text{m} = 6.38 \times 10^6\ \text{m}$$

So

$$V = \frac{4}{3}\pi \left(6.38 \times 10^6\ \text{m}\right)^3$$
$$= 1.09 \times 10^{21}\ \text{m}^3 \text{ to three significant figures.}$$

The Earth's volume is $1.09 \times 10^{21}\ \text{m}^3$.

Question 3.15 (b)

From Equation 3.18

$$F_g = G\,\frac{m_1 m_2}{r^2}$$

$$G = 6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$$

$$m_1 = 5.97 \times 10^{24}\ \mathrm{kg}$$

$$m_2 = 7.35 \times 10^{22}\ \mathrm{kg}$$

$$\begin{aligned} r &= 3.84 \times 10^5\ \mathrm{km} \\ &= 3.84 \times 10^5 \times 10^3\ \mathrm{m} \\ &= 3.84 \times 10^8\ \mathrm{m} \end{aligned}$$

Substituting values into the equation gives

$$F_g = 6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}} \times \frac{5.97 \times 10^{24}\ \mathrm{kg} \times 7.35 \times 10^{22}\ \mathrm{kg}}{(3.84 \times 10^8\ \mathrm{m})^2}$$

Rearranging to collect the units together

$$F_g = \frac{6.673 \times 10^{-11} \times 5.97 \times 10^{24} \times 7.35 \times 10^{22}\ \mathrm{N\,m^2\,kg^{-2}\,kg\,kg}}{(3.84 \times 10^8)^2\ \mathrm{m^2}}$$

Many of the units can be cancelled

$$F_{\mathrm{g}} = \frac{6.673 \times 10^{-11} \times 5.97 \times 10^{24} \times 7.35 \times 10^{22}\ \mathrm{N}\ \cancel{\mathrm{m}^2}\ \cancel{\mathrm{kg}^{-2}}\ \cancel{\mathrm{kg}}\ \cancel{\mathrm{kg}}}{(3.84 \times 10^{8})^2\ \cancel{\mathrm{m}^2}}$$

Calculating the numeric value gives

$F_{\mathrm{g}} = 1.99 \times 10^{20}$ N to 3 significant figures.

{Note that there was no need to express the newtons in terms of base units on this occasion; all the other units cancelled to leave N as the units of force, as expected.}

The magnitude of the gravitational force between the Earth and the Moon is 1.99×10^{20} N.

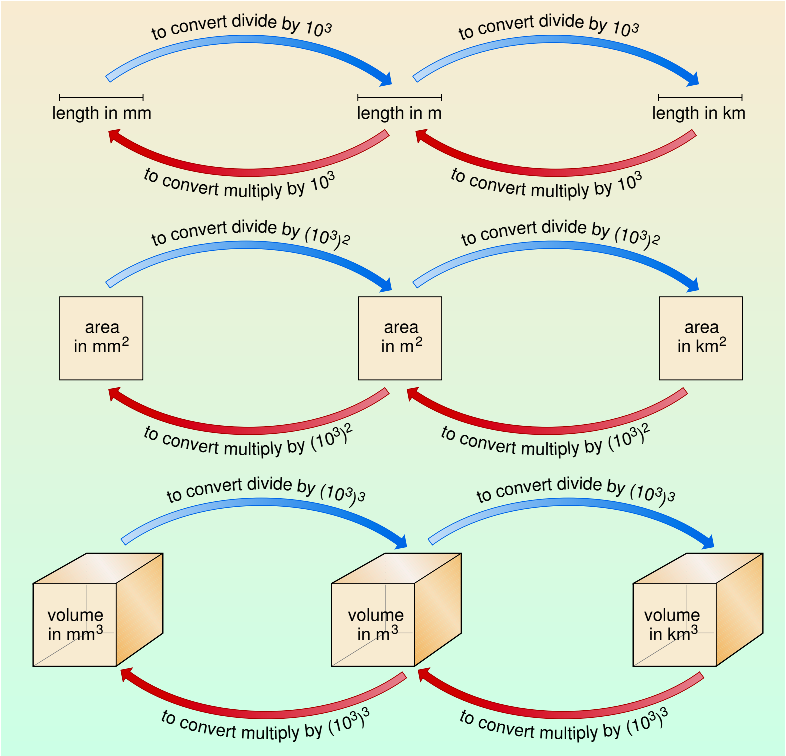


Figure 3.8: Unit conversions for length, area and volume.

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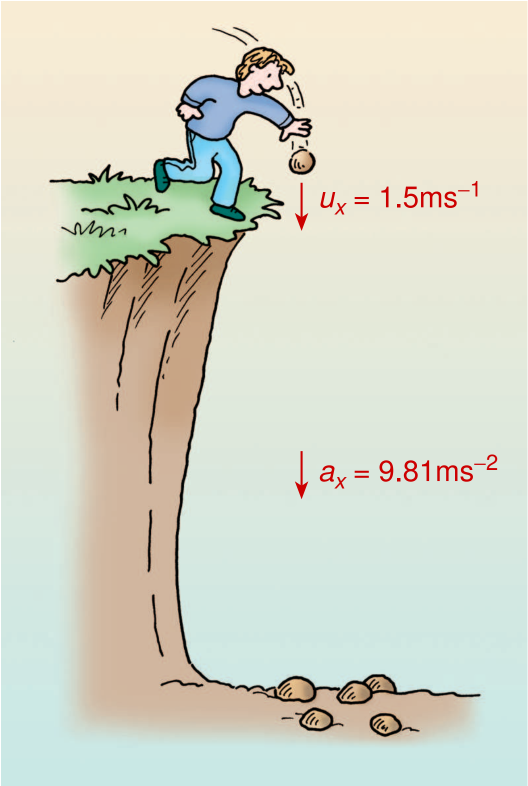

Figure 3.11: A stone being thrown from a cliff.

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Box 3.4 Some scientific formulae

$$C = 2\pi r \tag{3.3}$$

where C is the circumference of a circle of radius r.

$$A = \pi r^2 \tag{3.4}$$

where A is the area of a circle of radius r.

$$V = \frac{4}{3}\pi r^3 \tag{3.5}$$

where V is the volume of a sphere of radius r.

$$F = ma \tag{3.6}$$

where F is the magnitude of force on an object, m is its mass and a is the magnitude of its acceleration.

$$E = mc^2 \tag{3.7}$$

where E is energy, m is mass and c is the speed of light.

$$GPP = NPP + R \tag{3.8}$$

where GPP is the gross primary production of energy by plants in an ecosystem, NPP is net primary production and R is energy used in plant respiration.

$$\rho = \frac{m}{V} \tag{3.9}$$

where ρ is the density of an object of mass m and volume V.

$$v_\mathrm{S} = \sqrt{\frac{\mu}{\rho}} \tag{3.10}$$

where v_S is the speed of an S wave travelling through rocks of density ρ and rigidity modulus μ.

$$P = \rho g h \tag{3.11}$$

where P is the pressure at depth h in a liquid of density ρ, and g is the acceleration due to gravity.

$$PV = nRT \tag{3.12}$$

where P is the pressure of n moles of a gas in a container of volume V held at temperature T and R is a constant called the gas constant.

$$v = f\lambda \tag{3.13}$$

where v is the speed of a wave, f is its frequency and λ is its wavelength.

$$q = mc\,\Delta T \tag{3.14}$$

where q is the heat transferred to an object, m is its mass, c is its specific heat capacity and ΔT is the change in its temperature.

$$v_{\mathrm{av}} = \frac{v_{\mathrm{i}} + v_{\mathrm{f}}}{2} \tag{3.15}$$

where v_{av} is average speed, v_{i} is initial speed and v_{f} is final speed.

$$v_x = u_x + a_x t \tag{3.16}$$

where u_x, v_x and a_x are respectively initial speed, final speed and acceleration, all in the direction of the x-axis, and t is time.

$$s_x = u_x t + \frac{1}{2} a_x t^2 \tag{3.17}$$

where s_x, u_x and a_x are respectively distance, initial speed and acceleration, all in the direction of the x-axis, and t is time.

$$F_{\mathrm{g}} = G\,\frac{m_1 m_2}{r^2} \tag{3.18}$$

where F_{g} is the magnitude of the gravitational force between two objects of masses m_1 and m_2, a distance r apart. G is a constant called Newton's universal gravitational constant.

$$v_{\mathrm{esc}} = \left(\frac{2GM}{R}\right)^{1/2} \tag{3.19}$$

where v_{esc} is the escape speed, i.e. the speed with which an object must be fired from the surface of a planet of mass M and radius R in order just to escape from it. G is Newton's universal gravitational constant.

$$d = [L/\,(4\pi\,F)]^{1/2} \tag{3.20}$$

where d is the distance at which light from a star of luminosity L has a flux density of F.

Return to Section 3.5.2

alpha	A	α	nu (new)	N	ν
beta	B	β	xi (csi)	Ξ	ξ
gamma	Γ	γ	omicron	O	o
delta	Δ	δ	pi (pie)	Π	π
epsilon	E	ϵ	rho (roe)	P	ρ
zeta	Z	ζ	sigma	Σ	σ
eta	H	η	tau (taw)	T	τ
theta	Θ	θ	upsilon	Υ	υ
iota	I	ι	phi (fie)	Φ	ϕ
kappa	K	κ	chi (kie)	X	χ
lambda	Λ	λ	psi	Ψ	ψ
mu (mew)	M	μ	omega	Ω	ω

Table 3.1: The Greek alphabet. The pronunciation is given in parentheses where it is not obvious.

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Question 4.1 (a)

$v = f\lambda$ can be reversed to give $f\lambda = v$.

To isolate f we need to remove λ, and f is currently *multiplied* by λ so, according to Hint 3, we need to *divide* by λ. Remember that we must do this to *both sides of the equation*, so we have

$$\frac{f\lambda}{\lambda} = \frac{v}{\lambda}$$

The λ in the numerator of the fraction on the left-hand side cancels with the λ in the denominator to give

$$f = \frac{v}{\lambda}$$

Question 4.1 (b)

E_{tot} = can be reversed to give $E_k + E_p = E_{tot}$.

To isolate E_k we need to remove E_p, and E_p is currently *added* to E_k so, according to Hint 1, we need to *subtract* E_p. Remember that we must do this to *both sides of the equation*, so we have

$$E_k + E_p - E_p = E_{tot} - E_p$$

$E_p - E_p = 0$, so

$$E_k = E_{tot} - E_p$$

Question 4.1 (c)

$\rho = \dfrac{m}{V}$ can be reversed to give $\dfrac{m}{V} = \rho$

To isolate m we need to remove V, and m is currently *divided* by V so, according to Hint 4, we need to *multiply* by V. Remember that we must do this to *both sides of the equation*, so we have

$$\frac{mV}{V} = \rho V$$

The V in the numerator of the fraction on the left-hand side cancels with the V in the denominator to give

$$m = \rho V$$

Question 4.2 (a)

$b = c - d + e$ can be written as $c - d + e = b$ (with e on the left-hand side).

Adding d to both sides gives

$$c - d + e + d = b + d$$

i.e.

$$c + e = b + d$$

Subtracting c from both sides gives

$$c + e - c = b + d - c$$

i.e.

$$e = b + d - c.$$

Question 4.2 (b)

$p = \rho g h$ can be written as $\rho g h = p$ (with h on the left-hand side).

Dividing both sides by ρ gives

$$\frac{\rho g h}{\rho} = \frac{p}{\rho}$$

i.e.

$$gh = \frac{p}{\rho}$$

Dividing both sides by g gives

$$\frac{gh}{g} = \frac{p}{\rho g}$$

i.e.

$$h = \frac{p}{\rho g}$$

Question 4.2 (c)

$$v_{\mathrm{esc}}^2 = \frac{2GM}{R}$$

Multiplying both sides by R (to get R onto the left-hand side) gives

$$\begin{aligned} v_{\mathrm{esc}}^2 R &= \frac{2GMR}{R} \\ &= 2GM \end{aligned}$$

Dividing both sides by v_{esc}^2 gives

$$\frac{v_{\mathrm{esc}}^2 R}{v_{\mathrm{esc}}^2} = \frac{2GM}{v_{\mathrm{esc}}^2}$$

i.e.

$$R = \frac{2GM}{v_{\mathrm{esc}}^2}$$

Question 4.2 (d)

$$E = hf - \phi$$

Adding ϕ to both sides (to get ϕ onto the left-hand side) gives

$$E + \phi = hf - \phi + \phi$$

i.e.

$$E + \phi = hf$$

Subtracting E from both sides gives

$$E + \phi - E = hf - E$$

that is

$$\phi = hf - E$$

Question 4.2 (e)

We need to start by finding an equation for c^2.

$a = \dfrac{bc^2}{d}$ can be written as $\dfrac{bc^2}{d} = a$ (with c on the left-hand side).

Multiplying both sides by d gives

$$\frac{bc^2 d}{d} = ad$$

i.e.

$$bc^2 = ad$$

Dividing both sides by b gives

$$\frac{bc^2}{b} = \frac{ad}{b}$$

i.e.

$$c^2 = \frac{ad}{b}$$

Taking the square root of both sides gives

$$c = \pm\sqrt{\frac{ad}{b}}$$

Question 4.2 (f)

$a = \sqrt{\dfrac{b}{c}}$ can be written as $\sqrt{\dfrac{b}{c}} = a$ (with b on the left-hand side)

Squaring both sides gives

$$\frac{b}{c} = a^2$$

Multiplying both sides by c gives

$$\frac{bc}{c} = a^2 c$$

i.e.

$$b = a^2 c$$

Question 4.3 (a)

We need to start by finding an equation for v^2.

$E_k = \frac{1}{2}mv^2$ can be written as $\frac{1}{2}mv^2 = E_k$. (with the v^2 on the left-hand side).

Multiplying both sides by 2 gives

$$mv^2 = 2E_k$$

Dividing both sides by m gives

$$v^2 = \frac{2E_k}{m}$$

Taking the square root of both sides gives

$$v = \pm\sqrt{\frac{2E_k}{m}}$$

but we are only interested in the positive value on this occasion.

Question 4.3 (b)

If $E_\mathrm{k} = 2 \times 10^3$ J and $m = 4 \times 10^{21}$ kg

$$\begin{aligned}
v &= \sqrt{\frac{2E_\mathrm{k}}{m}} \\
&= \sqrt{\frac{2 \times 2 \times 10^3\ \mathrm{J}}{4 \times 10^{21}\ \mathrm{kg}}} \\
&= \sqrt{1 \times 10^{-18}\ \frac{\cancel{\mathrm{kg}}\,\mathrm{m}^2\,\mathrm{s}^{-2}}{\cancel{\mathrm{kg}}}} \\
&= 1 \times 10^{-9}\ \mathrm{m\,s^{-1}}
\end{aligned}$$

{At this speed, the plate would move 3 cm in a year.}

Question 4.3 (c)

If $E_\mathrm{k} = 2 \times 10^3$ J and $m = 70$ kg

$$\begin{aligned} v &= \sqrt{\frac{2E_\mathrm{k}}{m}} \\ &= \sqrt{\frac{2 \times 2 \times 10^3\ \mathrm{J}}{70\ \mathrm{kg}}} \\ &= 8\ \mathrm{m\,s^{-1}} \end{aligned}$$

{The sprinter, having a smaller mass, has to move rather faster than the tectonic plate!}

Question 4.4 (a)

$v_x = u_x + a_x t$ can be written as

$$u_x + a_x t = v_x$$

Subtracting u_x from both sides gives

$$a_x t = v_x - u_x$$

Dividing both sides by t gives

$$a_x = \frac{v_x - u_x}{t}$$

Question 4.4 (b)

Squaring both sides of $v_\mathrm{s} = \sqrt{\dfrac{\mu}{\rho}}$ gives

$$v_s^2 = \frac{\mu}{\rho}$$

Multiplying both sides by ρ gives

$$\rho\, v_s^2 = \mu$$

Dividing both sides by v_s^2 gives

$$\rho = \frac{\mu}{v_s^2}$$

Question 4.4 (c)

Multiplying both sides of $F = \dfrac{L}{4\pi d^2}$ by d^2 gives

$$Fd^2 = \frac{L}{4\pi}$$

Dividing both sides by F gives

$$d^2 = \frac{L}{4\pi F}$$

Taking the square root of both sides gives

$$d = \pm\sqrt{\frac{L}{4\pi F}}$$

{Note that if we consider just the positive value, we have arrived at Equation 3.20, albeit written rather differently.}

Question 4.5 (a)

$$\frac{\mu_0}{2\pi} \times \frac{i_1 i_2}{d} = \frac{\mu_0 \times i_1 i_2}{2\pi \times d} = \frac{\mu_0 i_1 i_2}{2\pi d}$$

Question 4.5 (b)

Note that $\frac{3a}{2b} \Big/ 2$ means $\frac{3a}{2b}$ divided by 2.

$$\frac{3a}{2b} \Big/ 2 = \frac{3a}{2b} \times \frac{1}{2} = \frac{3a}{4b}$$

Question 4.5 (c)

The product $c \times b$ will be a common denominator, so we can write

$$\frac{2b}{c} + \frac{3c}{b} = \frac{2b \times b}{c \times b} + \frac{3c \times c}{b \times c} = \frac{2b^2 + 3c^2}{cb}$$

This is the simplest form in which this fraction can be expressed.

Question 4.5 (d)

$$\frac{2ab}{c} \div \frac{2ac}{b} = \frac{2ab}{c} \times \frac{b}{2ac}$$

Cancelling the ‘2a’s gives

$$\frac{2ab}{c} \div \frac{2ac}{b} = \frac{\cancel{2a}b}{c} \times \frac{b}{\cancel{2a}c} = \frac{b^2}{c^2}$$

{Note that, for all parts of Question 4.5 and for many other questions involving simplification, it is possible to check that the algebraic expression you end up with is equivalent to the one that you started with by substituting numerical values for the variables. For example, setting $a = 2$, $b = 3$ and $c = 4$ in the original expression gives

$$\begin{aligned}\frac{2ab}{c} \div \frac{2ac}{b} &= \left(\frac{2 \times 2 \times 3}{4}\right) \div \left(\frac{2 \times 2 \times 4}{3}\right)\\ &= \frac{12}{4} \div \frac{16}{3} = 3 \div \frac{16}{3} = 3 \times \frac{3}{16} = \frac{9}{16}\end{aligned}$$

Substituting the same values in the answer gives $\dfrac{b^2}{c^2} = \dfrac{3^2}{4^2} = \dfrac{9}{16}$ }

Question 4.5 (e)

The product $f(f+1)$ will be a common denominator, so we can write

$$\begin{aligned}\frac{1}{f} - \frac{1}{f+1} &= \frac{(f+1)}{f(f+1)} - \frac{f}{(f+1)f} \\ &= \frac{f+1-f}{f(f+1)} \\ &= \frac{1}{f(f+1)}\end{aligned}$$

Question 4.5 (f)

$$\begin{aligned}\frac{2b^2}{(b+c)} \div \frac{2c^2}{(a+c)} &= \frac{\cancel{2}b^2}{(b+c)} \times \frac{(a+c)}{\cancel{2}c^2}\\ &= \frac{b^2(a+c)}{c^2(b+c)}\end{aligned}$$

The expression can be written as $\left(\frac{b}{c}\right)^2 \frac{(a+c)}{(b+c)}$ but cannot be simplified further.

Question 4.6

The equation can be written as

$$\begin{aligned}\frac{1}{f} &= \frac{1}{u} + \frac{1}{v}\\ &= \frac{v}{uv} + \frac{u}{vu} \quad \text{(taking the product } uv \text{ as the common denominator)}\\ &= \frac{v+u}{uv}\end{aligned}$$

Taking the reciprocal of both sides of the equation gives

$$f = \frac{uv}{v+u}$$

Question 4.7 (a)

$$\frac{1}{2}(v_x + u_x)\,t = \frac{1}{2}v_x t + \frac{1}{2}u_x t$$

or alternatively

$$\frac{1}{2}(v_x + u_x)\,t = \frac{v_x t}{2} + \frac{u_x t}{2} \text{ or } \frac{v_x t + u_x t}{2}$$

Question 4.7 (b)

$$\begin{aligned}\frac{(a-b)-(a-c)}{2} &= \frac{a-b-a+c}{2}\\ &= \frac{c-b}{2}\end{aligned}$$

since $a - a = 0$, and $-b + c$ is more tidily written as $c - b$.

Question 4.7 (c)

$$
\begin{aligned}
(k-2)(k-3) &= k^2 - 3k - 2k + 6 \\
&= k^2 - 5k + 6
\end{aligned}
$$

Question 4.7 (d)

$$
\begin{aligned}
(t-2)^2 &= (t-2)(t-2) \\
&= t^2 - 2t - 2t + 4 \\
&= t^2 - 4t + 4
\end{aligned}
$$

Question 4.8 (a)

$$y^2 - y = y(y-1)$$

Question 4.8 (b)

$x^2 - 25 = (x+5)(x-5)$, by comparison with Equation 4.3.

We can check that the factorization is correct by multiplying the brackets out. This gives

$$\begin{aligned}(x+5)(x-5) &= x^2 - 5x + 5x - 25 \\ &= x^2 - 25\end{aligned}$$

Question 4.9

Both the terms on the right-hand side of $E_{\text{tot}} = \frac{1}{2}mv^2 + mg\,\Delta h$ include m, so we can rewrite the equation as

$$E_{\text{tot}} = m\left(\tfrac{1}{2}v^2 + g\,\Delta h\right)$$

Reversing the order gives

$$m\left(\tfrac{1}{2}v^2 + g\,\Delta h\right) = E_{\text{tot}}$$

Dividing both sides by $\left(\frac{1}{2}v^2 + g\,\Delta h\right)$ gives

$$m = \frac{E_{tot}}{\frac{1}{2}v^2 + g\,\Delta h}$$

This is a perfectly acceptable equation for m, but the fraction in the denominator looks a little untidy. Multiplying the numerator and denominator by 2 gives

$$m = \frac{2E_{tot}}{v^2 + 2g\,\Delta h}$$

Question 4.10 (a)

From the answer to Question 4.7 (c)

$k^2 - 5k + 6 = (k - 2)(k - 3)$

Thus, if $k^2 - 5k + 6 = 0$, then $(k - 2)(k - 3) = 0$ too,
so $k - 2 = 0$ or $k - 3 = 0$.
i.e. $k = 2$ or $k = 3$

Checking for $k = 2$:
$k^2 - 5k + 6 = 2^2 - (5 \times 2) + 6 = 4 - 10 + 6 = 0$, as expected.
Checking for $k = 3$:
$k^2 - 5k + 6 = 3^2 - (5 \times 3) + 6 = 9 - 15 + 6 = 0$, as expected.

So the solutions of the equation $k^2 - 5k + 6 = 0$ are $k = 2$ and $k = 3$.

Question 4.10 (b)

From the answer to Question 4.7 (d)

$t^2 - 4t + 4 = (t - 2)^2$

Thus, if $t^2 - 4t + 4 = 0$, then $(t - 2)^2 = 0$ too,
so $t - 2 = 0$,
i.e. $t = 2$.

Checking:
$t = 2$ gives $t^2 - 4t + 4 = 2^2 - (4 \times 2) + 4 = 4 - 8 + 4 = 0$, as expected.

So the solution of the equation $t^2 - 4t + 4 = 0$ is $t = 2$.

Question 4.10 (c)

Comparison of $k^2 - 5k + 6 = 0$ with $ax^2 + bx + c = 0$ shows that $a = 1$, $b = -5$ and $c = 6$ on this occasion, so the solutions are

$$\begin{aligned} k &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-5) \pm \sqrt{(-5)^2 - (4 \times 1 \times 6)}}{2 \times 1} \\ &= \frac{5 \pm \sqrt{25 - 24}}{2} \\ &= \frac{5 \pm 1}{2} \end{aligned}$$

so $k = \dfrac{5+1}{2} = \dfrac{6}{2} = 3$ or $k = \dfrac{5-1}{2} = \dfrac{4}{2} = 2$.

So the solutions of the equation $k^2 - 5k + 6 = 0$ are $k = 2$ and $k = 3$. This is the same answer as was obtained in part (a) and could be checked in the same way.

Question 4.10 (d)

Comparison of $t^2 - 4t + 4 = 0$ with $ax^2 + bx + c = 0$ shows that $a = 1$, $b = -4$ and $c = 4$ on this occasion, so the solutions are

$$\begin{aligned}
k &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-4) \pm \sqrt{(-4)^2 - (4 \times 1 \times 4)}}{2 \times 1} \\
&= \frac{4 \pm \sqrt{16 - 16}}{2} \\
&= \frac{4 \pm 0}{2} \\
&= 2
\end{aligned}$$

So there is just one solution to $t^2 - 4t + 4 = 0$; namely $t = 2$. This is the same answer as was obtained in part (b) and could be checked in the same way.

Question 4.11 (a)

Rearranging $p = mv$ to make v the subject gives

$$v = \frac{p}{m} \quad \text{(dividing both sides by } m\text{)}$$

Substituting in $E_{\mathrm{k}} = \frac{1}{2}mv^2$ gives

$$\begin{aligned} E_{\mathrm{k}} &= \tfrac{1}{2}m\left(\frac{p}{m}\right)^2 \\ &= \tfrac{1}{2}m\frac{p^2}{m^2} \\ &= \frac{p^2}{2m} \end{aligned}$$

Question 4.11 (b)

Since both equations are already written with E (the variable we are trying to eliminate) as the subject, we can simply set the two equations for E equal to each other:

$$\tfrac{1}{2}mv^2 = mg\,\Delta h$$

There is an m on both sides of the equation; dividing both sides of the equation by m gives

$$\tfrac{1}{2}v^2 = g\,\Delta h$$

Multiplying both sides of the equation by 2 gives

$$v^2 = 2g\,\Delta h$$

Taking the square root of both sides of the equation gives

$$v = \pm\sqrt{2g\,\Delta h}$$

Question 4.11 (c)

Rearranging $c = f\lambda$ to make f the subject gives

$$f = \frac{c}{\lambda} \quad \text{(dividing both sides by } \lambda\text{)}$$

Substituting in $E_\mathrm{k} = hf - \phi$ gives

$$E_\mathrm{k} = \frac{hc}{\lambda} - \phi$$

Adding ϕ to both sides of the equation gives

$$E_\mathrm{k} + \phi = \frac{hc}{\lambda}$$

Subtracting E_k from both sides gives

$$\phi = \frac{hc}{\lambda} - E_\mathrm{k}$$

Question 4.12

Let the number selected be represented by x:

Adding 5 gives	$x + 5$
Doubling the result gives	$2(x + 5) = 2x + 10$
Subtracting 2 gives	$(2x + 10) - 2 = 2x + 8$
Dividing by 2 gives	$\dfrac{2x + 8}{2} = x + 4$

Taking away the number you first thought of gives $(x + 4) - x = 4$.

Question 4.13

Let H represent Helen's height in cm and T represent Tracey's height in cm. Since Tracey is 15 cm taller than Helen we can write

$$T = H + 15 \qquad \text{(i)}$$

The height of the wall is equal to Tracey's height up to her shoulders $(T - 25)$ plus Helen's height up to her eyes $(H - 10)$, thus

$$(T - 25) + (H - 10) = 300 \qquad \text{(ii)}$$

Simplifying (ii) gives

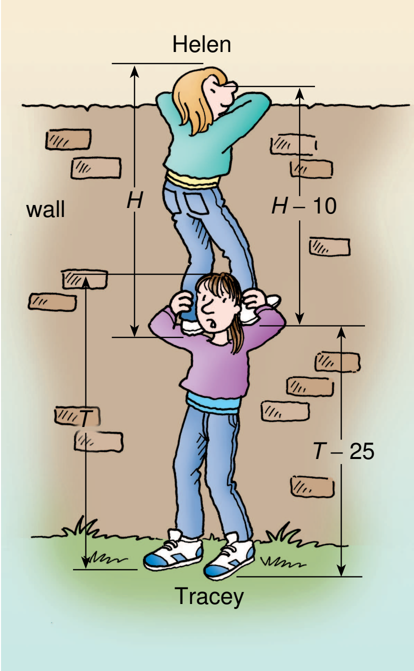
$$T + H - 35 = 300$$

Adding 35 to both sides gives

$$T + H = 335$$

Substituting for T from (i) gives

$$(H + 15) + H = 335$$
$$2H + 15 = 335$$



Subtracting 15 from both sides gives

$$2H = 320$$

Dividing both sides by 2 gives

$$H = 160$$

i.e. Helen is 160 cm tall.

Question 4.14

The equations required are $E_{\mathrm{g}} = mg\,\Delta h$ (Equation 4.18) and $E_{\mathrm{k}} = \frac{1}{2}mv^2$ (Equation 4.17).

Assuming that the child's gravitational potential energy is converted into kinetic energy, $E_{\mathrm{k}} = E_{\mathrm{g}}$.

$$\tfrac{1}{2}mv^2 = mg\,\Delta h$$

Dividing both sides by m gives

$$\tfrac{1}{2}v^2 = g\,\Delta h$$

Multiplying both sides by 2 gives

$$v^2 = 2g\,\Delta h$$

Taking the square root of both sides gives

$$v = \pm\sqrt{2g\,\Delta h}$$

On this occasion we are only interested in the positive square root, i.e. $v = \sqrt{2g\,\Delta h}$

Substituting $\Delta h = 1.8$ m and $g = 9.81\ \mathrm{m\,s^{-2}}$ gives

$$\begin{aligned} v &= \sqrt{2 \times 9.81\ \mathrm{m\,s^{-2}} \times 1.8\ \mathrm{m}} \\ &= 5.9\ \mathrm{m\,s^{-1}} \text{ to two significant figures} \end{aligned}$$

(noting that $\sqrt{\mathrm{m^2\,s^{-1}}} = \mathrm{m\,s^{-1}}$).

Checking

The units have worked out to be $\mathrm{m\,s^{-1}}$, as expected.

An estimated value is

$$\begin{aligned} v &\approx \sqrt{2 \times 10\ \mathrm{m\,s^{-2}} \times 2\ \mathrm{m}} \\ &\approx \sqrt{40\ \mathrm{m^2\,s^{-2}}} \\ &\approx 6\ \mathrm{m\,s^{-1}}, \text{ since } \sqrt{40} \approx \sqrt{36} \end{aligned}$$

The speed seems quite high; in reality not all of the child's gravitational potential energy would be converted into kinetic energy.

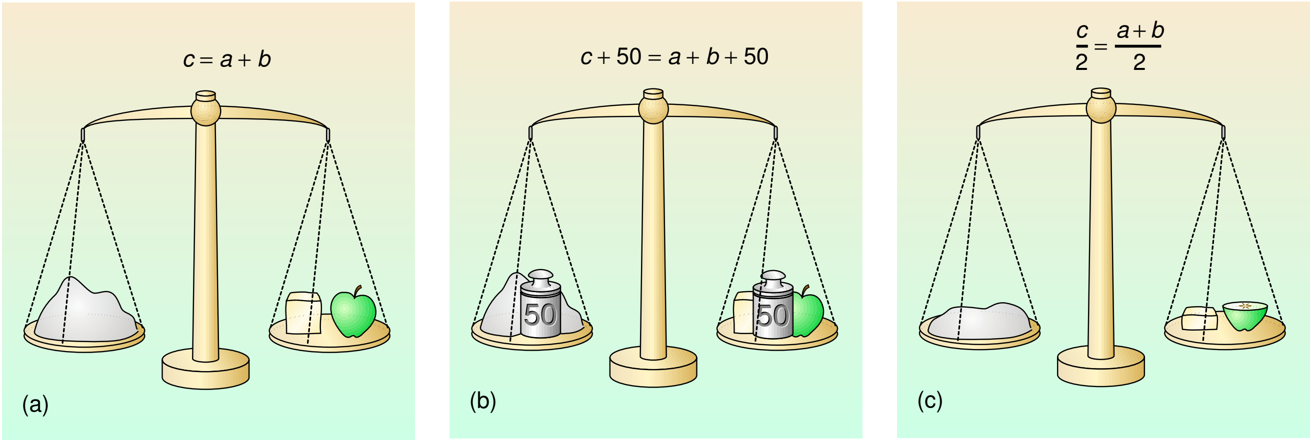


Figure 4.1: (a) The analogy between an equation and a set of kitchen scales. The scales remain balanced if (b) 50 g is added to both sides or if (c) the weight on both sides is halved.

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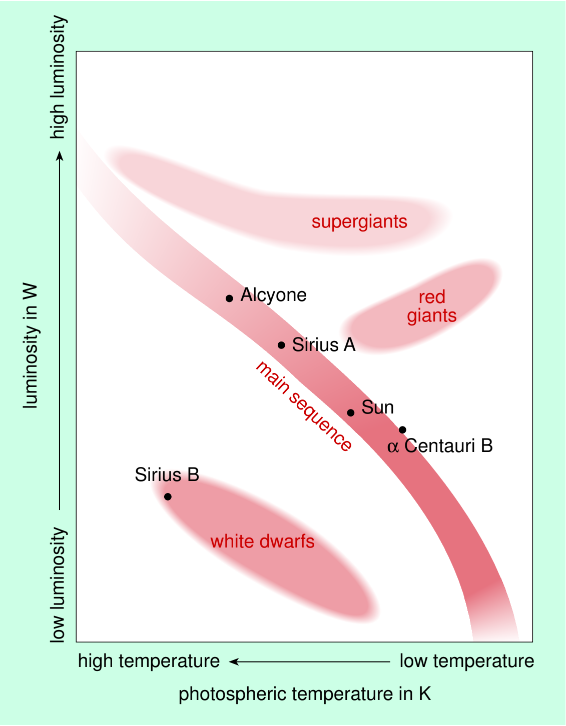

Figure 4.2: A Hertzsprung–Russell diagram showing the Sun and a number of other stars.

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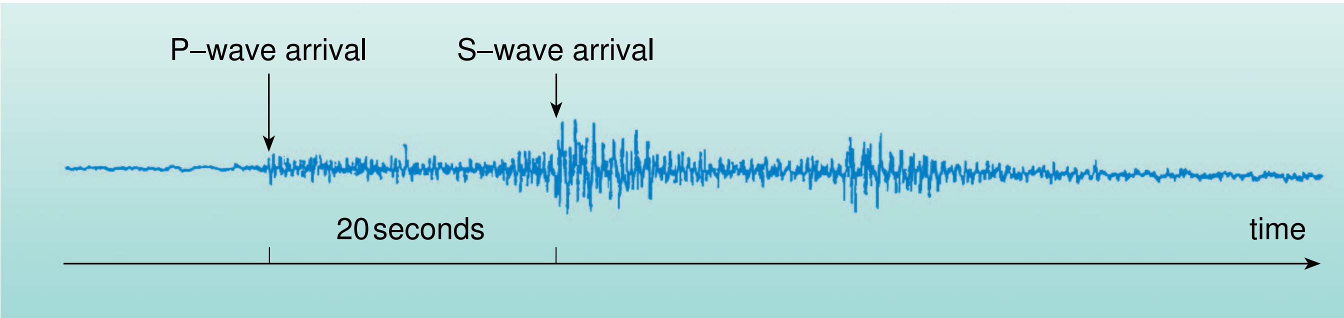

Figure 4.4: Seismogram recorded at the British Geological Survey in Edinburgh on 12 September 1988 at 2.23 p.m.

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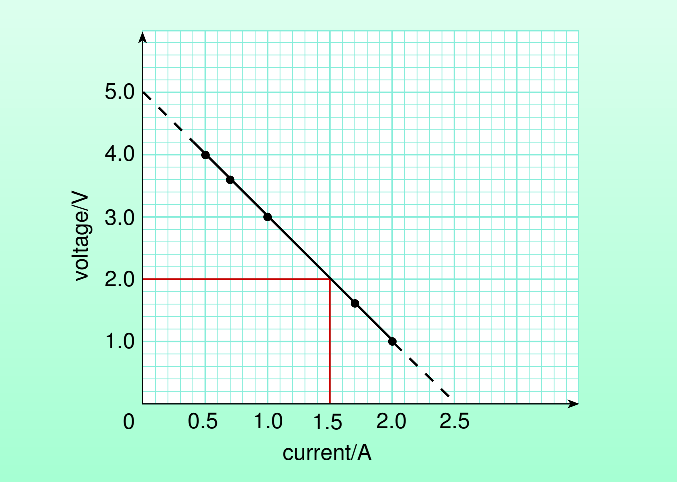
Question 5.1

(a) The red lines on the graph show that, by interpolation, when current = 1.5 A then voltage = 2.0 V.

(b) The line through the data points can be extended at each end, as shown below. This process of extrapolation to the vertical axis shows that when the current is zero the voltage has a value of 5.0 V.

(c) Extrapolation to the horizontal axis shows that when the voltage is zero the current has a value of 2.5 A.



Question 5.2

Using the red lines on Figure 5.38,

$$\text{gradient} = \frac{\text{rise}}{\text{run}}$$
$$= \frac{(170 - 10)\text{ km}}{(32 - 4)\text{ s}}$$
$$= \frac{160\text{ km}}{28\text{ s}}$$
$$= 5.7\text{ km s}^{-1}$$

Therefore the average speed of the P waves is 5.7 km s^{-1} (to two significant figures).

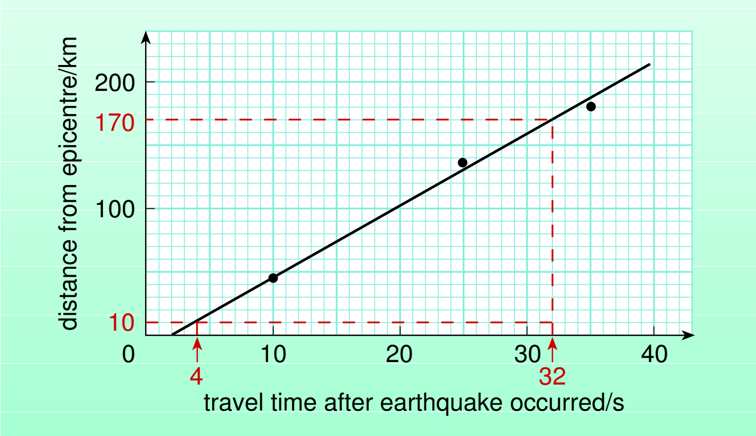


Figure 5.38

{You may have chosen different points from which to calculate your gradient, but you should still have got the same answer. Note that the scale of the graph does not really allow points to be specified to more than two significant figures, so this is the precision to which the answer should be given.}

Question 5.3

Using the red lines on Figure 5.39,

$$\begin{aligned}\text{gradient} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{(32-2)\ \text{s}}{(170-0)\ \text{km}} \\ &= \frac{30\ \text{s}}{170\ \text{km}} \\ &= 0.176\ \text{s}\,\text{km}^{-1}\end{aligned}$$

Therefore

$$\begin{aligned}\text{speed} &= \frac{1}{0.176\ \text{s}\,\text{km}^{-1}} \\ &= 5.7\ \text{km}\,\text{s}^{-1}\ \text{(to two significant figures)}\end{aligned}$$

To the precision to which it is possible to read the graph, this is the same value as before.

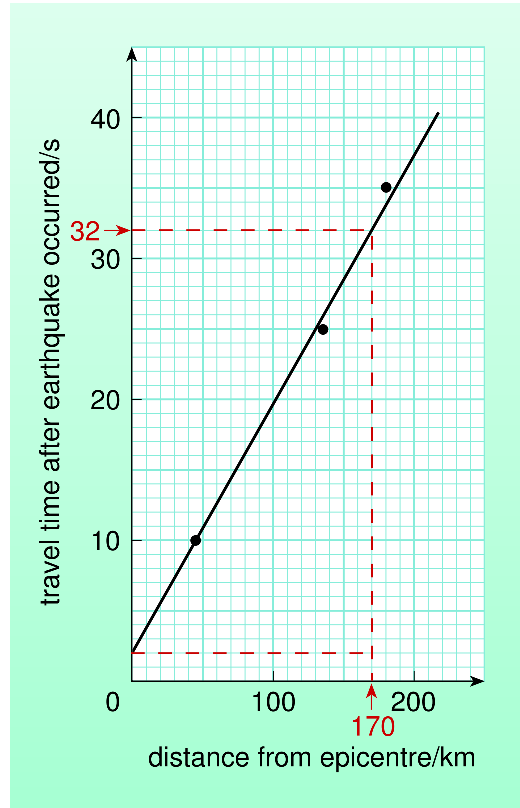


Figure 5.39

Question 5.4

Using the red lines on Figure 5.40,

$$\begin{aligned} \text{gradient} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{(7-20)\ ^\circ\text{C}}{(2-0)\ \text{km}} \\ &= \frac{-13\ ^\circ\text{C}}{2\ \text{km}} \\ &= -6.5\ ^\circ\text{C}\,\text{km}^{-1} \end{aligned}$$

This could also have been written as $-6.5\ ^\circ\text{C/km}$. The negative value of the gradient implies that temperature decreases with increasing height above sea-level and your sentence should reflect this. For example you could write: 'For each successive kilometre of height gained above sea-level, the atmospheric temperature falls by 6.5 °C'.

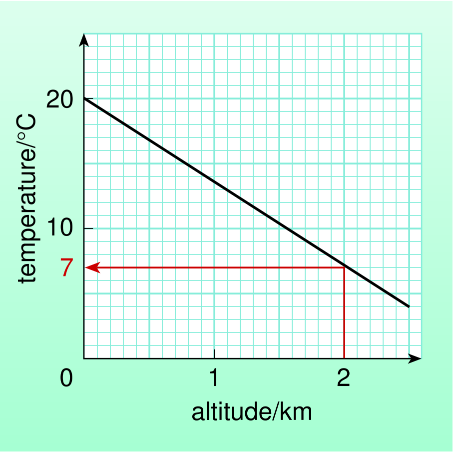


Figure 5.40

Question 5.5

Using the red lines on Figure 5.41,

$$\begin{aligned} \text{gradient} &= \frac{(-50 - (-20))\ \ ^\circ\text{C}}{(10 - 5.5)\ \text{km}} \\ &= \frac{-30\ ^\circ\text{C}}{4.5\ \text{km}} \\ &= -6.7\ ^\circ\text{C}\,\text{km}^{-1} \end{aligned}$$

This agrees quite well with the value obtained in the answer to Question 5.4. In fact temperature does decrease with altitude at an almost constant rate through the troposphere.

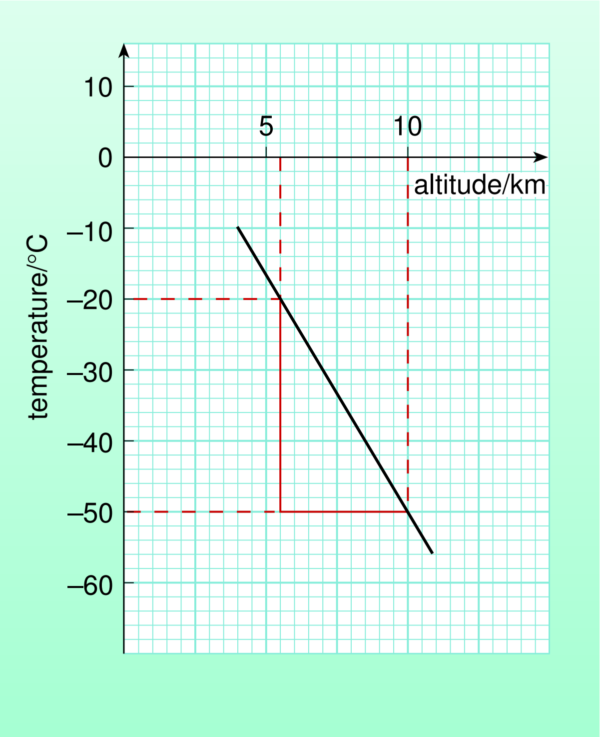
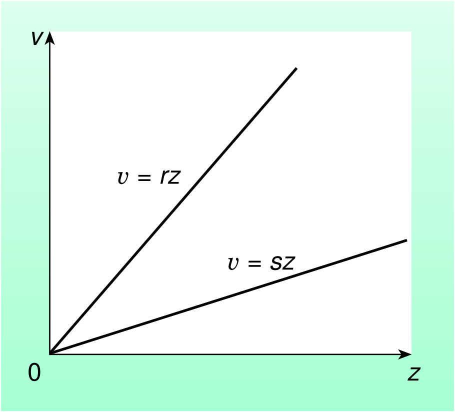


Figure 5.41

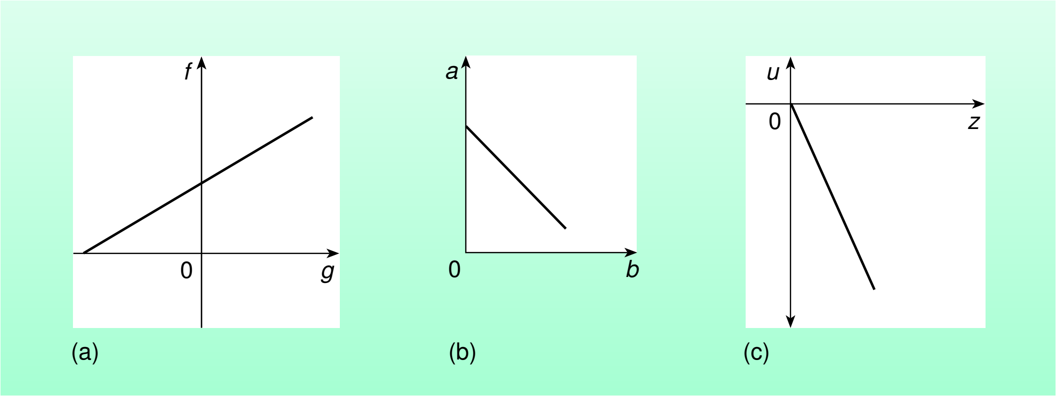
Question 5.6

The line corresponding to $v = rz$ has the larger (steeper) gradient. Therefore r is larger than s.



Question 5.7

If two quantities are directly proportional to each other, a graph in which one is plotted against the other will be a straight line *through the origin*. Therefore, only (c) corresponds to a proportional relationship: $u \propto z$. In this case, the gradient is negative, i.e. the constant of proportionality is negative.



Question 5.8

Since M is directly proportional to d^3, these are the quantities to plot. The spheres are selected and then their masses are measured, so d is the independent variable, and so according to convention d^3 should therefore be plotted on the horizontal axis. In other words, the convention would be to plot M against d^3.

Slightly rearranging the equation and comparing with the standard equation of a straight line

$$\begin{array}{ccccccc} y & = & m & x & + & c & \\ M & = & \dfrac{\pi\rho}{6} & d^3 & (+ & 0 &) \end{array}$$

shows that the gradient would be $\dfrac{\pi\rho}{6}$.

{If you chose to defy convention and plot d^3 against M, the gradient would have the reciprocal value, i.e. $\dfrac{6}{\pi\rho}$.}

Question 5.9

There are at least two equally valid ways to plot the data here. Since

$$T = 2\pi\sqrt{\frac{L}{g}}$$

squaring both sides gives

$$T^2 = \frac{4\pi^2 L}{g}$$

L is the independent variable, which according to convention should be plotted on the horizontal axis. A graph of T^2 against L has gradient $= \dfrac{4\pi^2}{g}$ so

$$g = \frac{4\pi^2}{\text{gradient}}$$

Alternatively, you could have chosen to plot T against $\sqrt{L}$. The gradient of this line would be $\dfrac{2\pi}{\sqrt{g}}$. So $\sqrt{g} = \dfrac{2\pi}{\text{gradient}}$ and

$$g = \frac{4\pi^2}{(\text{gradient})^2}$$

Question 5.10

After n half-lives, the number of radioactive atoms is reduced to $\left(\frac{1}{2}\right)^n$ of the original number.

Since $\dfrac{1}{16} = \left(\dfrac{1}{2}\right)^4$

four half-lives must elapse before the number of radioactive atoms will be $\frac{1}{16}$ of the number there are today. So 4×1600 years $= 6400$ years must elapse for this to happen.

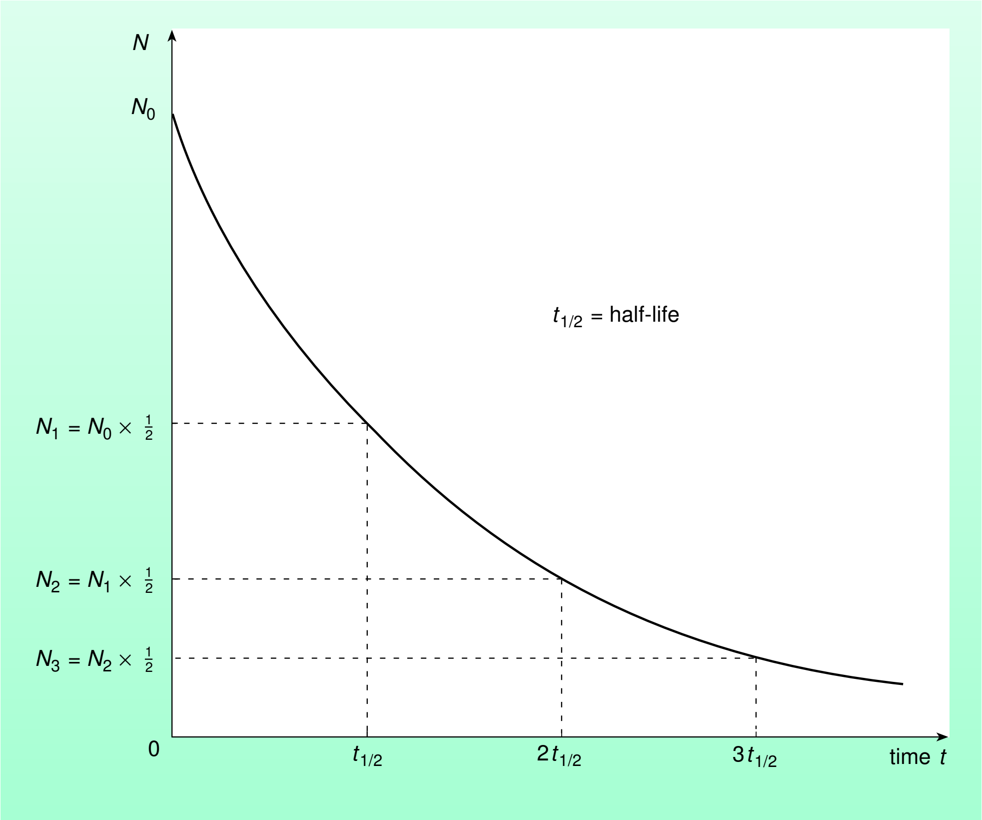


Figure 5.35: Radioactive decay.

N_0 radioactive nuclei are present at time $t = 0$. During each half-life, the number of radioactive nuclei is halved. The half-life is denoted by the symbol $t_{1/2}$.

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Box 5.3 Sea-floor spreading

Plate tectonics describes how the outer layer of the Earth is made up of a series of 'plates' which move relative to one another. The top layer of these plates is known as the 'crust'. Ocean crust is about 7 km thick, continental crust up to 80 km thick. The crust is split at mid-ocean ridges and material is ejected at the ridge crests to form new sea-floor. This creation of new crust is balanced at the opposite end of the plate by material being forced under an adjacent plate. As eruption at a mid-ocean ridge continues, older sea-floor crust is moved aside to make way for younger crust and the sea-floor 'spreads' symmetrically away from the ridge as shown in Figure 5.8a. Recently formed ocean crust is largely inaccessible, so scientists interested in the speed at which this spreading occurs have to resort to indirect means of measuring it.

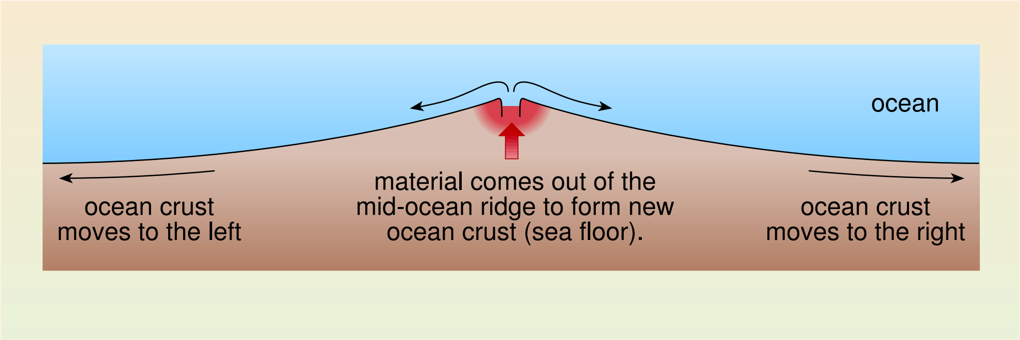


Figure 5.8a: Production of new sea-floor at a mid-ocean ridge.

A record of their age is held in rocks by their magnetism. The orientation of the Earth's magnetic field has reversed at irregular intervals throughout its history, and the orientation of the magnetic field at the time a rock was formed is 'locked into' the rock. The times at which these changes in orientation took place are known from measurements on a great many surface rocks that can be dated by a variety of means.

Figure 5.8b shows the timescale for reversals in the Earth's magnetic field over the last 4 Ma. Black denotes 'normal' polarity (i.e. what we experience today) and white denotes reversed polarity.

Marine magnetic surveys reveal patterns in the orientation of the magnetization of rocks near mid-ocean ridges; an idealized pattern is shown in Figure 5.8c. Correlation of patterns like this with the pattern in Figure 5.8b provides a method of dating the rocks at various distances from a mid-ocean ridge.

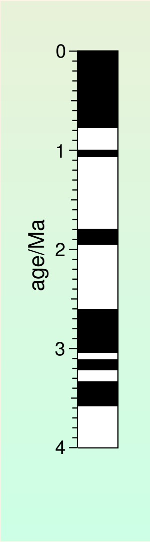


Figure 5.8b

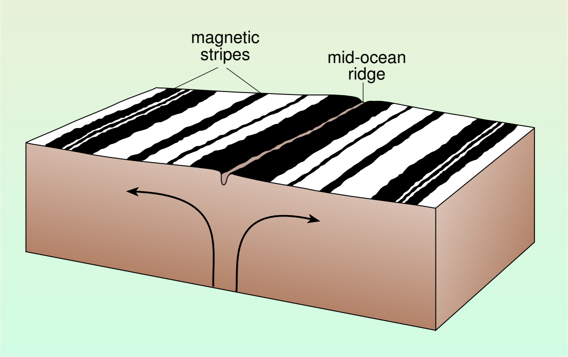


Figure 5.8c: Idealized symmetrical 'magnetic stripes' either side of a ridge; rocks shown black are magnetized in the opposite direction to those shown white and can be matched with the timescale of Figure 5.8b

Return to Section 5.2.1

Question 6.1 (a)

$$
\begin{aligned}
2\pi \text{ radians} &= 360^\circ \\
\text{so } 1 \text{ radian} &= \frac{360^\circ}{2\pi}
\end{aligned}
$$

$$
\begin{aligned}
0.123 \text{ radians} &= 0.123 \times \frac{360^\circ}{2\pi} \\
&\approx 7.05^\circ \text{ to three significant figures.}
\end{aligned}
$$

Question 6.1 (b)

2π radians $= 360°$

so $\dfrac{2\pi}{3}$ radians $= \dfrac{360°}{3} = 120°$

Question 6.1 (c)

2π radians $= 360°$

so π radians $= 180°$

$$\frac{3\pi}{2} \text{ radians} = \frac{3 \times 180°}{2} = \frac{540°}{2} = 270°$$

Question 6.2 (a)

$360^\circ = 2\pi$ radians

so $1^\circ = \dfrac{2\pi}{360}$ radians

$$\begin{aligned} 36.5^\circ &= 36.5 \times \frac{2\pi}{360} \text{ radians} \\ &\approx 0.637 \text{ radians to three significant figures.} \end{aligned}$$

Question 6.2 (b)

$360^\circ = 2\pi$ radians

so $1^\circ = \dfrac{2\pi}{360}$ radians

$$\begin{aligned} 90^\circ &= 90 \times \frac{2\pi}{360} \text{ radians} \\ &= \frac{\pi}{2} \text{ radians.} \end{aligned}$$

{This answer could have been written as 1.57 radians (to 3 significant figures), but note that $\dfrac{\pi}{2}$ radians is an exact answer which 1.57 radians is not.}

Question 6.2 (c)

$360^\circ = 2\pi$ radians

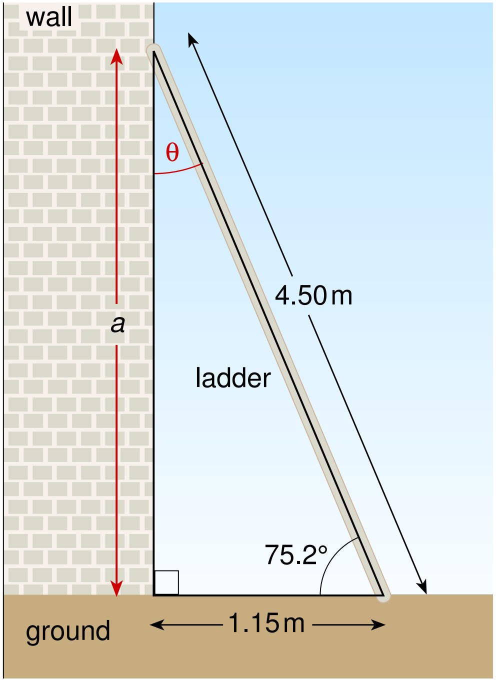
so $1^\circ = \dfrac{2\pi}{360}$ radians

$$\begin{aligned} 210^\circ &= 210 \times \frac{2\pi}{360} \text{ radians} \\ &= \frac{7\pi}{6} \text{ radians.} \end{aligned}$$

{This answer could have been written as 3.67 radians (to 3 significant figures), but note that $\dfrac{7\pi}{6}$ radians is an exact answer which 3.67 radians is not.}

Question 6.3 (a)

We are trying to find length a in the diagram.



From Pythagoras' Theorem

$$a^2 + (1.15\text{ m})^2 = (4.50\text{ m})^2$$

so

$$a^2 = (4.50\text{ m})^2 - (1.15\text{ m})^2$$

$$a = \sqrt{20.25\text{ m}^2 - 1.3225\text{ m}^2}$$

$$= 4.35\text{ m to three significant figures.}$$

Question 6.3 (b)

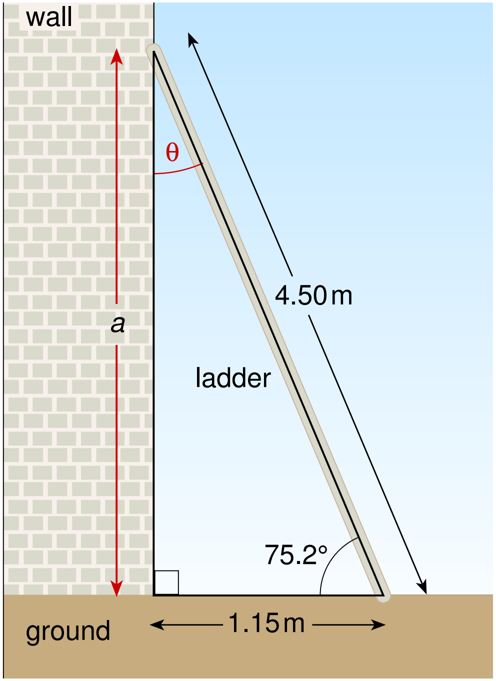
We are trying to find angle θ in the diagram.

The interior angles in a triangle add up to 180° so

$$\theta + 75.2^\circ + 90^\circ = 180^\circ$$

i.e.

$$\begin{aligned}\theta &= 180^\circ - 75.2^\circ - 90^\circ \\ &= 14.8^\circ.\end{aligned}$$



Question 6.4 (a)

$\sin 49^\circ = 0.7547$

Question 6.4 (b)

$$\cos\frac{\pi}{8} = 0.9239$$

{Since the angle was given in radians, your calculator needs to be in 'radians mode' in order to obtain the correct answer to this part.}

Question 6.4 (c)

$$\tan \frac{\pi}{4} = 1$$

{Since the angle was given in radians, your calculator needs to be in ‘radians mode’ in order to obtain the correct answer to this part.}

Question 6.5 (a)

$$\cos^{-1}(0.5253) = 58.31^\circ$$

{Your calculator needs to be in 'degrees mode' in order to obtain the correct answer.}

Question 6.5 (b)

$$\tan^{-1}(1.5574) = 1.0000 \text{ radians}$$

{Your calculator needs to be in ‘radians mode’ in order to obtain the correct answer.}

Question 6.6 (a)

$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$

so

$$\cos 32^\circ = \frac{4.3\ \text{m}}{h}$$

Multiplying both sides by h gives

$$h\cos 32^\circ = 4.3\ \text{m}$$

Dividing both sides by $\cos 32^\circ$ gives

$$\begin{aligned} h &= \frac{4.3\ \text{m}}{\cos 32^\circ} \\ &= 5.1\ \text{m to two significant figures.} \end{aligned}$$

Question 6.6 (b)

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

so

$$\sin\frac{\pi}{3} = \frac{a}{10\text{ m}}$$

Multiplying both sides by 10 m gives

$$\begin{aligned} a &= 10\text{ m} \times \sin\frac{\pi}{3} \\ &= 8.7\text{ m to two significant figures.} \end{aligned}$$

Question 6.6 (c)

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$= \frac{5.0\text{ m}}{1.0\text{ m}}$$
$$= 5.0$$

So $\theta = \tan^{-1}(5.0)$
$= 79^\circ$

{Note that ‘opp’ and ‘adj’ must be the sides opposite and adjacent to the angle you are trying to find.}

Question 6.7

$$
\begin{aligned}
H &= \text{height of West Tower} + \text{height of base of Cathedral} - \text{height of theodolite}\\
&= 66\text{ m} + 15\text{ m} - 1.5\text{ m}\\
&= 79.5\text{ m}
\end{aligned}
$$

$$
\theta = 2.27^\circ
$$

$$
\tan\theta = \frac{H}{D}
$$

Multiplying both sides by D gives

$$
D\tan\theta = H
$$

Dividing both sides by $\tan\theta$ gives

$$
\begin{aligned}
D &= \frac{H}{\tan\theta}\\
&= \frac{79.5\text{ m}}{\tan 2.27^\circ}\\
&= 2006\text{ m}\\
&\approx 2000\text{ m}
\end{aligned}
$$

So you can estimate the distance of the theodolite from Ely Cathedral to be about 2 km.

Question 6.8

From Equation 6.10,

$$V = W \tan\theta$$

where $W = 65$ m and $\theta = 36°$. So

$$\begin{aligned} V &= 65\ \text{m} \times \tan 36° \\ &= 47\ \text{m to two significant figures.} \end{aligned}$$

The vertical thickness of the stratum is 47 metres.

Question 6.9

From Equation 6.11,

$$r = h \cos 45^\circ$$

where r is the required radius and $h = 302$ pm. So

$$\begin{aligned} r &= 302\ \text{pm} \times \cos 45^\circ \\ &= 214\ \text{pm to three significant figures.} \end{aligned}$$

The radius of a lithium ion is 214 pm (i.e. 2.14×10^{-10} m)

Question 6.10

$i = 45.0°$ $\quad r = 26.3°$ $\quad v_1 = 3.00 \times 10^8\ \mathrm{m\,s^{-1}}$

Snell's law states that

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2}$$

We are trying to find v_2, the speed of light in glass.

Multiplying both sides of $\dfrac{\sin i}{\sin r} = \dfrac{v_1}{v_2}$ by v_2 and by $\sin r$ gives

$$v_2 \sin i = v_1 \sin r$$

Dividing both sides by $\sin i$ gives

$$\begin{aligned} v_2 &= v_1 \frac{\sin r}{\sin i} \\ &= 3.00 \times 10^8\ \mathrm{m\,s^{-1}} \times \frac{\sin 26.3°}{\sin 45.0°} \\ &= 3.00 \times 10^8\ \mathrm{m\,s^{-1}} \times \frac{0.4431}{0.7071} \\ &= 1.88 \times 10^8\ \mathrm{m\,s^{-1}} \end{aligned}$$

So the speed of light in glass is $1.88 \times 10^8\ \mathrm{m\,s^{-1}}$.

Question 6.11

From Equation 6.13

$$\sin\theta_n = \frac{n\lambda}{d}$$

Reversing the equation and multiplying both sides by d gives

$$n\lambda = d\sin\theta_n$$

Dividing both sides by n

$$\lambda = \frac{d\sin\theta_n}{n}$$

$d = 1.64 \times 10^{-6}$ m $\quad \theta_n = 24.1^\circ \quad n = 1$

So

$$\begin{aligned}\lambda &= \frac{1.64 \times 10^{-6}\ \text{m} \times \sin 24.1^\circ}{1} \\ &= 6.70 \times 10^{-7}\ \text{m to three significant figures.}\end{aligned}$$

Question 6.12

Let the distance to car ferry = d.
The length of car ferry = l = 86 m.
The angle subtended = θ = 3.5°.

Converting θ to radians:

$$360^\circ = 2\pi \text{ radians}$$
$$\text{so } 1^\circ = \frac{2\pi}{360} \text{ radians}$$
$$3.5^\circ = 3.5 \times \frac{2\pi}{360} \text{ radians} = 0.0611 \text{ radians}$$

From Equation 6.1

$$\theta \text{ (in radians)} = \frac{s}{r}$$

In this case $s \approx l$ and $r \approx d$ so

$$\theta \approx \frac{l}{d}$$

Multiplying both sides by d gives

$$\theta\, d \approx l$$

Dividing both sides by θ gives

$$d = \frac{l}{\theta}$$

So

$$
\begin{aligned}
d &\approx \frac{86\ \mathrm{m}}{0.0611} \\
&\approx 1408\ \mathrm{m}
\end{aligned}
$$

The ferry is approximately 1.4 km away.

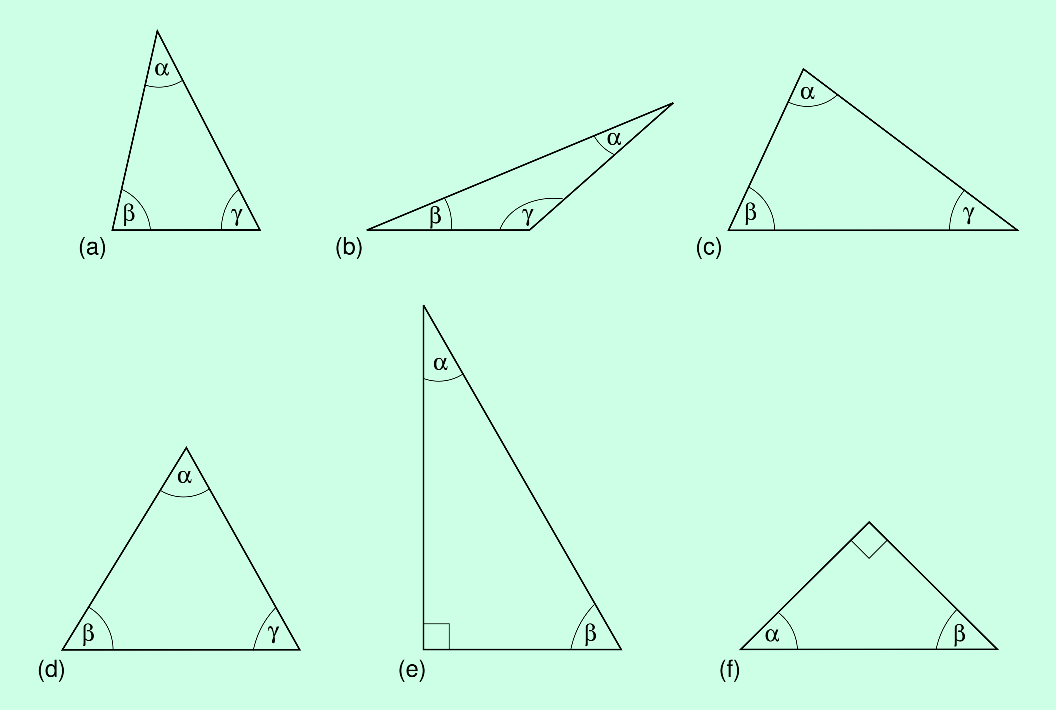


Figure 6.12: Triangles of various shapes.

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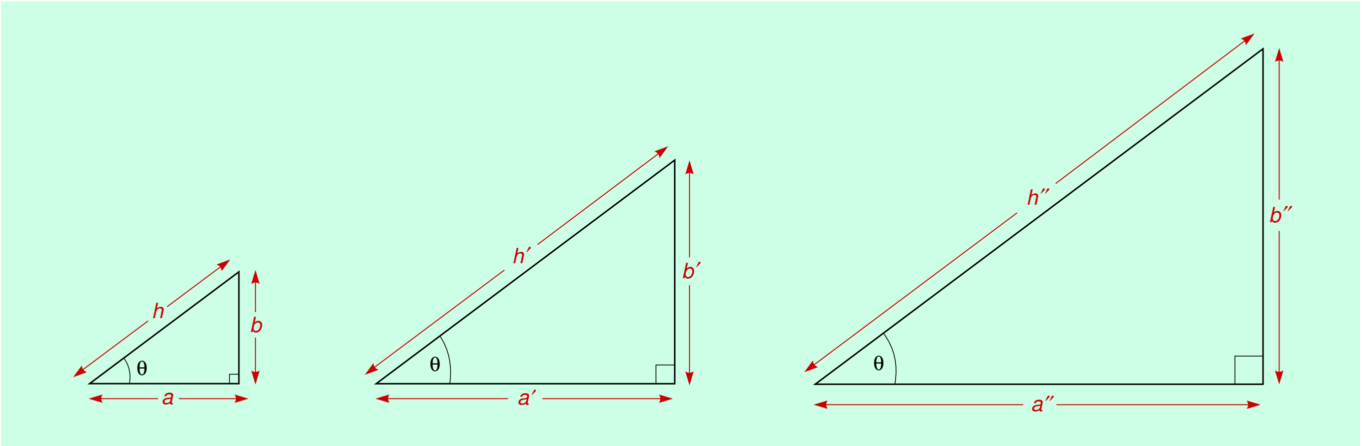

Figure 6.14: Three similar right-angled triangles.

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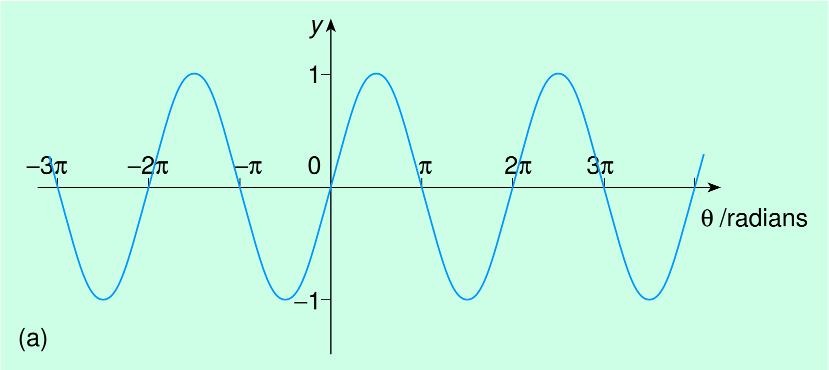

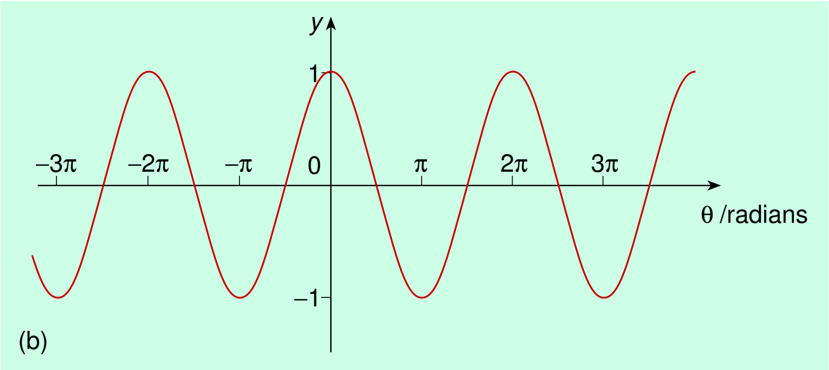

Figure 6.17: Graphs of (a) $y = a\sin\theta$, (b) $y = a\cos\theta$.

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Question 7.1 (a)

Since $100 = 10^2$, $\log_{10} 100 = 2$.

Question 7.1 (b)

Since $0.001 = 10^{-3}$, $\log_{10} 0.001 = -3$.

Question 7.1 (c)

Since $\sqrt{10} = 10^{1/2}$, $\log_{10} \sqrt{10} = \dfrac{1}{2}$

Question 7.1 (d)

Since $1.329 = 10^{0.1235}$ (from the section of text just above the question), $\log_{10} 1.329 = 0.1235$.

Question 7.2 (a)

$\log_{10} 2 = 0.3010$

Question 7.2 (b)

$\log_{10} 2000 = 3.301$

{ Note that $\log_{10} 2000$ is exactly 3 greater than $\log_{10} 2$. This result will be discussed further in Sections 7.2 and 7.3. }

Question 7.3 (a)

$10^{1.5} = 31.62$

Question 7.3 (b)

$p = 31.62$

{ Because of the way in which log to base 10 is defined, this follows straight from the answer to Question 7.3(a). }

Question 7.4 (a)

For human blood the hydrogen ion concentration is $4.0 \times 10^{-8}\ \mathrm{mol\,dm^{-3}}$, so

$$\begin{aligned}
\mathrm{pH} &= -\log_{10}\left(\frac{4.0 \times 10^{-8}\ \mathrm{mol\,dm^{-3}}}{\mathrm{mol\,dm^{-3}}}\right) \\
&= -\log_{10}\left(4.0 \times 10^{-8}\right) \\
&= -(-7.4) \\
&= 7.4
\end{aligned}$$

Question 7.4 (b)

For the hair shampoo, the hydrogen ion concentration is 3.2×10^{-6} mol dm^{-3}, so

$$\begin{aligned}
\text{pH} &= -\log_{10}\left(\frac{3.2 \times 10^{-6}\ \text{mol dm}^{-3}}{\text{mol dm}^{-3}}\right) \\
&= -\log_{10}\left(3.2 \times 10^{-6}\right) \\
&= -(-5.5) \\
&= 5.5
\end{aligned}$$

Question 7.5 (a)

$$
\begin{aligned}
\log_{10} 300 &= \log_{10}(3 \times 100) \\
&= \log_{10} 3 + \log_{10} 100 && \text{(from Equation 7.2)} \\
&= 0.4771 + \log_{10} 10^2 \\
&= 0.4771 + 2 && \text{(from Equation 7.1)} \\
&= 2.477 \text{ to four significant figures.}
\end{aligned}
$$

Question 7.5 (b)

$$\begin{aligned}
\log_{10} 0.03 &= \log_{10}(3 \div 100) \\
&= \log_{10} 3 - \log_{10} 100 && \text{(from Equation 7.3)} \\
&= 0.4771 - \log_{10} 10^2 \\
&= 0.4771 - 2 && \text{(from Equation 7.1)} \\
&= -1.523 \text{ to four significant figures.}
\end{aligned}$$

Question 7.5 (c)

$$\begin{aligned}
\log_{10} 9 &= \log_{10}\left(3^2\right) \\
&= 2\log_{10} 3 && \text{(from Equation 7.4)} \\
&= 2 \times 0.4771 \\
&= 0.9542 \text{ to four significant figures.}
\end{aligned}$$

Question 7.6

The gradient of the line $= \dfrac{2.5 - 0.5}{1.0 - 0.0} = \dfrac{2.0}{1.0} = 2.0.$

This is the result expected.

The intercept of the line on the vertical axis is approximately 0.5.

$\log_{10} \pi = 0.497$ to three significant figures, so the result seems reasonable.

Question 7.7

Taking the log to base 10 of both sides of the equation $y = 2x^3$ gives

$$\begin{aligned}\log_{10} y &= \log_{10}\left(2x^3\right)\\ &= \log_{10} 2 + \log_{10} x^3 \quad \text{(from Equation 7.2)}\\ &= \log_{10} 2 + 3\log_{10} x \quad \text{(from Equation 7.4)}\end{aligned}$$

We can reverse the order of the two terms on the right-hand side to give

$$\log_{10} y = 3\log_{10} x + \log_{10} 2$$

Comparison with the general equation of a straight-line graph, $y = mx + c$, reveals that $m = 3$ and $c = \log_{10} 2$, so the gradient of the graph will be 3 and the intercept on the vertical axis will be $\log_{10} 2$.

Question 7.8

$n = n_0\, e^{at}$

Taking the log to base 10 of both sides of the equation gives

$$\begin{aligned}\log_{10} n &= \log_{10}\left(n_0\, e^{at}\right)\\ &= \log_{10} n_0 + \log_{10} e^{at} \qquad \text{(from Equation 7.2)}\\ &= \log_{10} n_0 + at \log_{10} e \qquad \text{(from Equation 7.4)}\end{aligned}$$

We can reverse the order of the two terms on the right-hand side to give

$$\begin{aligned}\log_{10} n &= at \log_{10} e + \log_{10} n_0\\ &= (a \log_{10} e)\, t + \log_{10} n_0\end{aligned}$$

Comparison with the general equation of a straight-line graph, $y = mx + c$, shows that a graph of $\log_{10} n$ against t will be a straight line of gradient $a \log_{10} e$ and intercept on the vertical axis of $\log_{10} n_0$.

Question 7.9 (a)

$\ln 4 = 1.386$

Question 7.9 (b)

The number whose natural logarithm is 4 is $e^4 = 54.60$.

Question 7.10

$$n = n_0 \, e^{at}$$

Taking the log to base e of both sides of the equation gives

$$\begin{aligned} \ln n &= \ln\left(n_0 \, e^{at}\right) \\ &= \ln n_0 + \ln e^{at} \qquad \text{(from Equation 7.8)} \\ &= \ln n_0 + at \qquad \text{(from Equation 7.7)} \end{aligned}$$

We can reverse the order of the two terms on the right-hand side to give

$$\ln n = at + \ln n_0$$

Comparison with the general equation of a straight-line graph, $y = mx + c$, shows that a graph of $\ln n$ against t will be a straight line of gradient a and intercept on the vertical axis of $\ln n_0$.

Stage	Number	k value
maximum total number of eggs if all pairs bred and laid 3 eggs (maximum possible)	$N_0 = 72$	$k_1 = \log_{10}\left(\dfrac{N_0}{N_1}\right) = \log_{10}\left(\dfrac{72}{51}\right) = 0.1498$
maximum possible number of eggs from the 17 pairs that *did* breed	$N_1 = 51$	$k_2 = \log_{10}\left(\dfrac{N_1}{N_2}\right) = \log_{10}\left(\dfrac{51}{43}\right) = 0.0741$
actual number of eggs laid	$N_2 = 43$	$k_3 = \log_{10}\left(\dfrac{N_2}{N_3}\right) = \log_{10}\left(\dfrac{43}{16}\right) = 0.4293$
number of eggs that hatched	$N_3 = 16$	$k_4 = \log_{10}\left(\dfrac{N_3}{N_4}\right) = \log_{10}\left(\dfrac{16}{15}\right) = 0.0280$
number of chicks that fledged	$N_4 = 15$	$k_5 = \log_{10}\left(\dfrac{N_4}{N_5}\right) = \log_{10}\left(\dfrac{\quad}{\quad}\right) =$
number of owlets that survive to form pairs	$N_5 = 9$	$k_{total} = \log_{10}\left(\dfrac{N_0}{N_5}\right) = \log_{10}\left(\dfrac{72}{9}\right) = 0.9031$

Table 7.1: k-values for various stages in the breeding of 24 pairs of owls in Wytham Wood in 1952–1953

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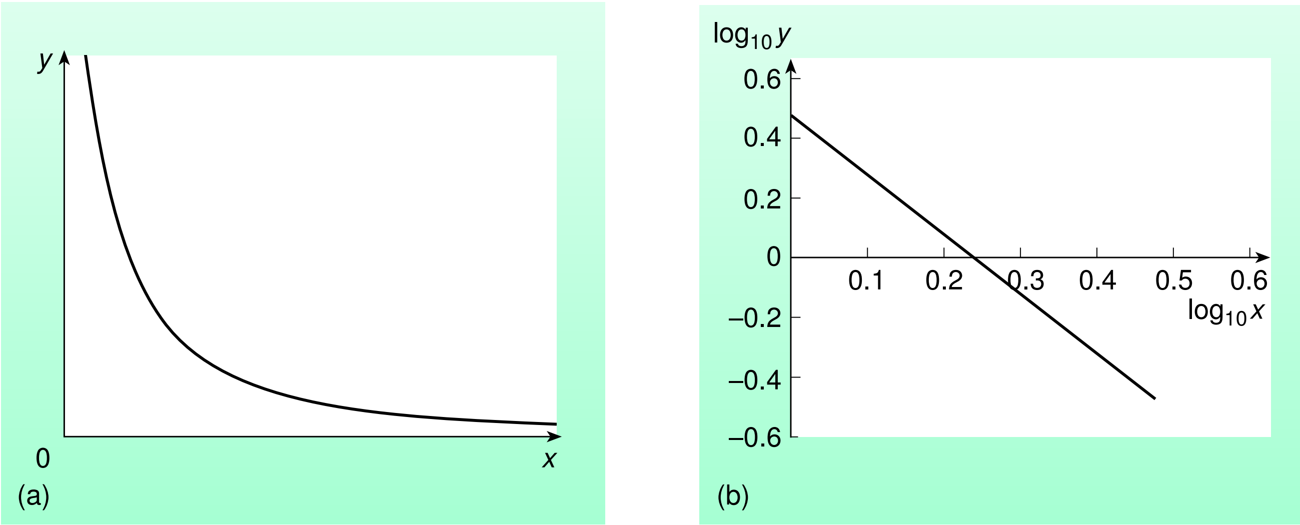

Figure 7.3: Graphs of (a) y against x and (b) $\log_{10} y$ against $\log_{10} x$ for the equation $y = 3x^{-2}$.

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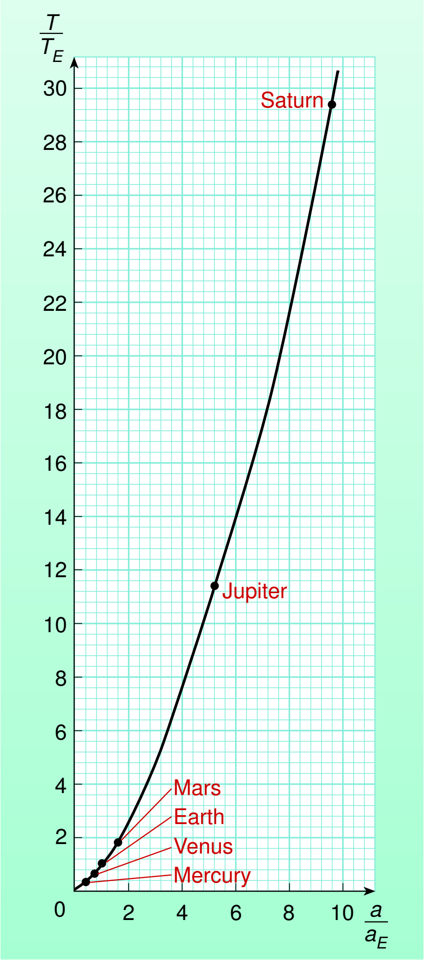

Figure 7.4: A graph of T/T_{E} against a/a_{E} where T is a planet's orbital period and a is the planet's average distance from the Sun. T_{E} and a_{E} are the values of T and a for the Earth, and the values of T and a for other planets have been divided by these so as to make the numbers plotted more manageable.

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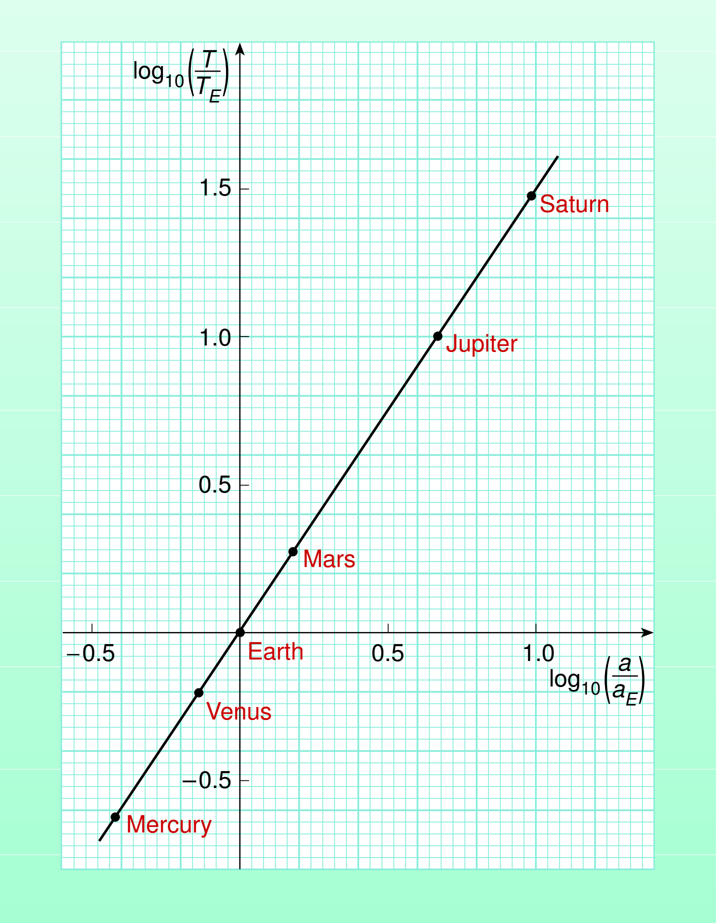

Figure 7.5: A graph of $\log_{10}(T/T_E)$ against $\log_{10}(a/a_E)$ where T is a planet's orbital period and a is the planet's average distance from the Sun. T_E and a_E are the values of T and a for the Earth.

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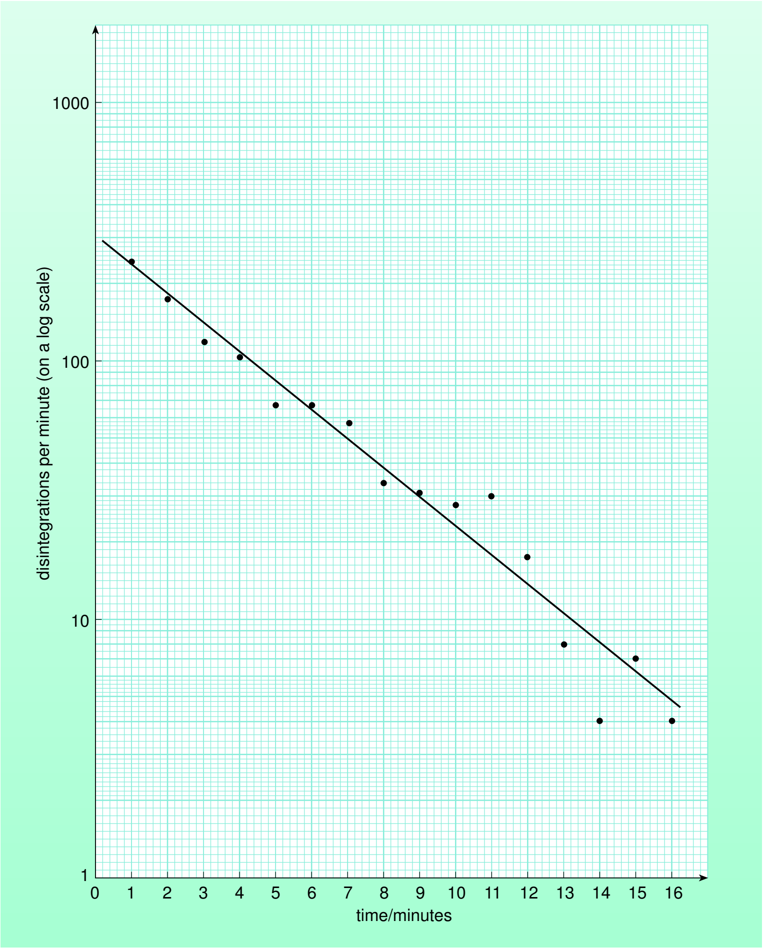

Figure 7.7: A graph of disintegrations per minute (on a log scale) against time for the radioactive decay of the excited state of barium-137.

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Question 8.1 (a)

Of the 52 cards in the pack, 13 are hearts. So according to Equation 8.2, the probability of a card drawn at random being a heart is $\frac{13}{52} = \frac{1}{4}$.

{This result also follows from noting that there are 4 suits, each with the same number of cards, so one-quarter will be hearts.}

Question 8.1 (b)

Of the 52 cards in the pack, 26 are red (13 hearts and 13 diamonds). So the probability of a card drawn at random being red is $\frac{26}{52} = \frac{1}{2}$.

{Or 2 of the 4 suits are red, so the probability is $\frac{2}{4} = \frac{1}{2}$.}

Question 8.1 (c)

Of the 52 cards in the pack, 4 are aces (one for each suit). So the probability of a card drawn at random being an ace is $\frac{4}{52} = \frac{1}{13}$.

Question 8.1 (d)

Of the 52 cards in the pack, 12 are picture cards (3 for each suit). So the probability of a card drawn at random being a picture card is $\frac{12}{52} = \frac{3}{13}$.

Question 8.2 (a)

For any one toss the probability of heads is always the same: $\frac{1}{2}$.

Question 8.2 (b)

For the single toss of the third coin, the probability of getting heads is $\frac{1}{2}$ and that is unaffected by what has gone before. This is no different to tossing the same coin three times in succession. Only foolish gamblers believe that because heads have come up twice running the chances of tails coming up the next time are thereby increased!

Question 8.3 (a)

If two coins are tossed simultaneously, there are four possible outcomes, all of which are equally likely:

Outcome 1	H	H
Outcome 2	H	T
Outcome 3	T	H
Outcome 4	T	T

The outcome of two tails can occur in only one way, so the probability of getting two tails is $\frac{1}{4}$.

This result can also be found from the multiplication rule:

the probability that the first coin will show tails is $\frac{1}{2}$;

the probability that the second coin will show tails is $\frac{1}{2}$;

so the probability of getting two tails is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Question 8.3 (b)

The probability of throwing a six with one dice is $\frac{1}{6}$. So the probability of getting a pair of sixes when throwing two dice is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

Question 8.4 (a)

Assuming the germination probabilities to be independent of one another, the probability of seeds of both A and B germinating is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

Question 8.4 (b)

Assuming the germination probabilities to be independent of one another, the probability of the seeds of all three species germinating is $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} = \frac{1}{24}$.

Question 8.4 (c)

Assuming the germination probabilities to be independent of one another,

the probability that a seed of A will *not* germinate is $\frac{1}{2}$;

the probability that a seed of B will *not* germinate is $\frac{2}{3}$;

the probability that a seed of C will *not* germinate is $\frac{3}{4}$;

so the probability that none will germinate is $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$.

Question 8.5

The probability of drawing any one particular card from the pack is $\frac{1}{52}$. This is true for each of the three named cards. So the probability of drawing the Jack of diamonds *or* the Queen of diamonds *or* the King of diamonds is $\frac{1}{52} + \frac{1}{52} + \frac{1}{52} = \frac{3}{52}$.

Question 8.6

The situation is similar to the one described in Question 8.3. If two coins are tossed simultaneously, there are four possible outcomes, all of which are equally likely:

Outcome 1	H	H
Outcome 2	H	T
Outcome 3	T	H
Outcome 4	T	T

The outcome of a head and a tail can occur in two ways, so the probability of getting a head and a tail is $\frac{2}{4} = \frac{1}{2}$.

This result can also be found from a combination of the multiplication and addition rules. For the combination of one head and one tail:

the probability that the coin on the left will be tails is $\frac{1}{2}$;

the probability that the coin on the right will be heads is $\frac{1}{2}$;

So the probability that the combination T H will occur is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

By similar reasoning, the probability that the combination H T will occur is also $\frac{1}{4}$. These possibilities are mutually exclusive, so the probability of getting one head and one tails is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Question 8.7

The fraction of the atmosphere that is oxygen is

$$\frac{0.26}{0.26 + 1} = \frac{0.26}{1.26}$$

Expressed as a percentage to 2 significant figures, this fraction is 21%.

Question 8.8

For the 10 measurements in Table 8.4,

mean = 1.122 mm

standard deviation s_n = 0.123 mm

Question 8.9 (a)

For nine measurements, the median is the 5^{th} measurement in the list (in ascending or descending order). This is 7.8 cm.

Question 8.9 (b)

From Equation 8.3, the mean is $\dfrac{70.4\ \text{cm}}{9} = 7.82\ \text{cm}$.

Question 8.10

The best estimate that can be made from this data of the mean number, μ, of flowers per plant in the colony is the mean of the sample, $\overline{x}$. In this case,

$$\overline{x} = 7.25 \text{ flowers}$$

{Note that it is normal practice to quote means and medians in this way, even for quantities, such as numbers of flowers, which cannot really be fractional!}

The best estimate that can be made of the standard deviation of the population is the sample standard deviation s_{n-1}. In this case,

$$s_{n-1} = 1.94 \text{ flowers.}$$

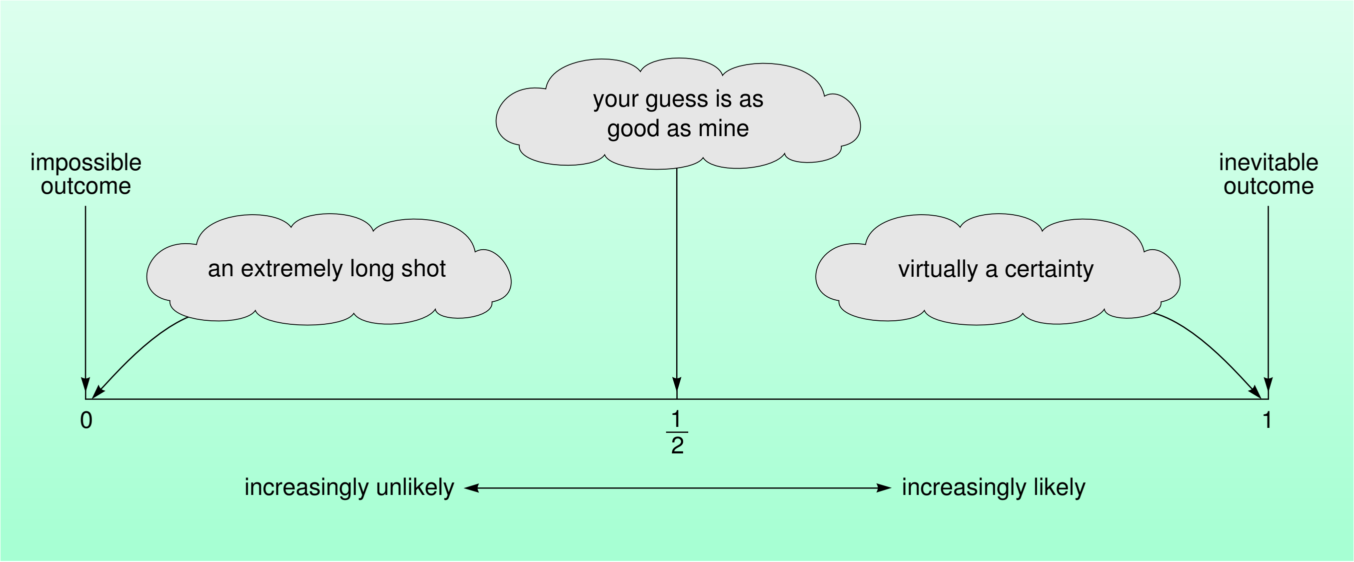

Figure 8.1: The scale of probabilities.

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x_i/nm	d_i/nm	$d_i^2/10^{-5}$ nm^2
2.458	0.0036	1.296
2.452	–0.0024	0.576
2.454	–0.0004	0.016
2.452	–0.0024	0.576
2.459	0.0046	2.116
2.455	0.0006	0.036
2.464	0.0096	9.216
2.453	–0.0014	0.196
2.449	–0.0054	2.916
2.448	–0.0064	4.096
$\sum_{i=1}^{n} x_i = 24.544$ nm	$\sum_{i=1}^{n} d_i = 0$	$\sum_{i=1}^{n} d_i^2 = 21.04 \times 10^{-5}$ nm^2
$\overline{x} = 2.4544$ nm		$\overline{d_i^2} = 2.104 \times 10^{-5}$ nm^2
		$s_n = \sqrt{\overline{d_i^2}}$
		$= 4.587 \times 10^{-3}$ nm
		$= 0.0046$ nm

Table 8.3: Calculation of the standard deviation for the set of measurements originally given in Table 8.2.

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Question 9.1 (a)

The answer to this question depends on which significance level is used. Employing the usual convention, i.e. rejecting the null hypothesis if $P < 0.05$, the null hypothesis should be rejected on this occasion, since $P < 0.01$ means that P must be less than 0.05. Therefore the alternative hypothesis should be accepted. However, if it had been decided only to reject the null hypothesis if P were less than 0.001, we would not be justified in categorically rejecting the null hypothesis in this way.

Question 9.1 (b)

Employing the usual convention, the null hypothesis should be accepted, since $P > 0.05$.

Question 9.1 (c)

This inequality is written in a way that is very unhelpful and ought to be avoided. We are told that $P > 0.01$. But how much greater? If $P > 0.05$ then, employing the usual convention, the null hypothesis must be accepted. However, if P lies between 0.05 and 0.01 (i.e. $0.05 > P > 0.01$) then, employing the usual convention, the null hypothesis should be rejected and the alternative hypothesis accepted. In the former situation, the result ought to have been given as $P > 0.05$; in the latter it ought to have been given as either $P < 0.05$ or $0.05 > P > 0.01$.

Question 9.2 (a)

Since the actual number of parasites per sheep is known, this data is at the interval level.

Question 9.2 (b)

Since the sheep are classified into just two contrasting categories (‘parasitized’ and ‘unparasitized’) this data is best treated as being at the categorical level.

{Since there is an element of ranking here, you might have regarded this data as being at the ordinal level. However, whether ‘unparasitized’ is ‘good’ or ‘bad’ does depend on whether you take the point of view of the sheep or the parasites! ‘Parasitized’ and ‘unparasitized’ *might* correspond to the clear-cut categories ‘susceptible to parasites’ and ‘resistant to parasites’. In general, ordinal level data is subdivided into more than two classes.}

Question 9.2 (c)

Since degree of parasitization is recorded, but not precisely how many parasites there were on each sheep, this data is at the ordinal level.

Question 9.3

The total number of plants in the next generation was 636 (i.e. 185 + 305 + 146). If the ratio in a sample of 636 plants were 1 red-flowered : 2 pink-flowered : 1 white-flowered, then there would be

$$\frac{636}{4} = 159 \text{ red-flowered plants}$$
$$\frac{636}{2} = 318 \text{ pink-flowered plants}$$
$$\frac{636}{4} = 159 \text{ white-flowered plants.}$$

These are therefore the ‘expected’ numbers. Drawing up a table, calculating each $\dfrac{(O_i - E_i)^2}{E_i}$ value and then summing these values to obtain the test statistic, χ^2:

Flower colour	O_i	E_i	$(O_i - E_i)$	$(O_i - E_i)^2$	$\dfrac{(O_i - E_i)^2}{E_i}$
red	185	159	26	676	4.252
pink	305	318	–13	169	0.531
white	146	159	–13	169	1.063
total	636	636	0		5.846

The number of degrees of freedom is given by

$$\begin{aligned}&\begin{pmatrix}\text{number of cells containing}\\ \text{observed numbers}\end{pmatrix} - 1\\ &\quad = 3 - 1\\ &\quad = 2\end{aligned}$$

Reading across the row for 2 degrees of freedom in Table 9.3, it can be seen that the χ^2 value of 5.846 corresponds to a significance level (P) of less than 0.1 but more than 0.05 (i.e. $0.1 > P > 0.05$).

Since the probability that the entire population from which the sample of 636 plants was drawn was in the ratio 1 red-flowered : 2 pink-flowered : 1 white-flowered is *greater than* 0.05, the null hypothesis cannot be rejected at the 5% significance level. The experimental data is therefore compatible with the prediction from genetics theory that the ratio of plants in the next generation should be in the ratio 1 red-flowered : 2 pink-flowered : 1 white-flowered.

Question 9.4

The values of $(R_A)_i$, $(R_B)_i$, D_i, D_i^2 and $\sum_{i=1}^{n} D_i^2$ are given below.

Vertical distance/cm	Rank $(R_A)_i$	Mean water content/ % dry mass	$(R_B)_i$	$D_i = (R_A)_i - (R_B)_i$	D_i^2
0	1	76	1	0	0
4	2	83	3	−1	1
7	3.5	93	4	−0.5	0.25
9	5	80	2	3	9
7	3.5	102	6	−2.5	6.25
11	7	95	5	2	4
10	6	120	7	−1	1
13	8	130	8	0	0
				$\sum_{i=1}^{n} D_i = 0$	$\sum_{i=1}^{n} D_i^2 = 21.5$

{Since there are two vertical distances of 7 cm, both are given the rank $\frac{3+4}{2} = 3.5$ and the next vertical distance (9 cm) is given the rank 5.}

Substituting $\sum_{i=1}^{n} D_i^2 = 21.5$ and $n = 8$ into Equation 9.2 gives

$$r_s = 1 - \frac{6 \times 21.5}{8 \times (8^2 - 1)} = 0.744$$

Reading across the row for 8 pairs of measurements in Table 9.8, it can be seen that $0.05 > P > 0.01$. Since $P < 0.05$, the null hypothesis must be rejected at the 5% significance level and the alternative hypothesis accepted. There is a statistically significant positive correlation between mean soil water content and vertical distance from ridge crest.

{Although mean soil water content was significantly correlated with both horizontal and vertical distance from the nearest ridge crest, the former produced a value of r_s that was both higher and more significant than the latter (i.e. $r_s = 0.905$, $P < 0.01$ compared to $r_s = 0.744$, $P < 0.05$). This was because horizontal distance from the ridge crest had been arranged to increase regularly.}

Question 9.5 (a)

These samples are unmatched. There is no logical link between any particular plant growing in one reserve and any particular plant growing in the other reserve.

Question 9.5 (b)

These samples are matched. For each sampling station along the stream, the number of nymphs is known for two species of Stonefly.

Question 9.6

In this case $\overline{x}_1 = 7.7$, $\overline{x}_2 = 7.2$, $s_1 = 2.7$, $s_2 = 2.1$, $n_1 = 18$ and $n_2 = 15$.

Substituting for s_1, s_2, n_1 and n_2 into Equation 9.5 gives

$$\begin{aligned}(S_c)^2 &= \frac{(18-1)(2.7)^2 + (15-1)(2.1)^2}{(18-1)+(15-1)} \\ &= \frac{(17 \times 7.29) + (14 \times 4.41)}{17+14} \\ &= 5.989\end{aligned}$$

Substituting for $(S_c)^2$, n_1 and n_2 into Equation 9.4 gives

$$SE_D = \sqrt{\frac{5.989}{18} + \frac{5.989}{15}} = 0.856$$

Substituting for $\overline{x}_1$, $\overline{x}_2$ and SE_D into Equation 9.3 gives

$$t = \frac{7.7 - 7.2}{0.856} = 0.584 \text{ to three places of decimals.}$$

In this case, the number of degrees of freedom is

$$(18-1) + (15-1) = 31.$$

The value of t (i.e. 0.584) is smaller than *any* of the critical values in the row for 30 degrees of freedom (the nearest equivalent to 31) in Table 9.13. The probability of obtaining a value of t as large as this by chance if the null hypothesis were true is therefore greater than 0.1 (i.e. $P > 0.1$), probably much greater. The difference in number of flowers per plant growing either side of this ridge is not statistically significant.

{Note: If you worked to a different number of significant figures in this question you may have obtained a slightly different value for t. However, your conclusion — that the difference in number of flowers per plant growing either side of this ridge is not statistically significant — should be have been the same.}

Degrees of freedom	$P = 0.1$	$P = 0.05$	$P = 0.01$	Degrees of freedom	$P = 0.1$	$P = 0.05$	$P = 0.01$
1	2.706	3.841	6.635	16	23.542	26.296	32.000
2	4.605	5.991	9.210	17	24.769	27.587	33.409
3	6.251	7.815	11.341	18	25.989	28.869	34.805
4	7.779	9.488	13.277	19	27.204	30.144	36.191
5	9.236	11.070	15.086	20	28.412	31.410	37.566
6	10.645	12.592	16.812	21	29.615	32.671	38.932
7	12.017	14.067	18.475	22	30.813	33.924	40.289
8	13.362	15.507	20.090	23	32.007	35.172	41.638
9	14.684	16.919	21.666	24	33.196	36.415	42.980
10	15.987	18.307	23.209	25	34.382	37.652	44.314
11	17.275	19.675	24.725	26	35.563	38.885	45.642
12	18.549	21.026	26.217	27	36.741	40.113	46.963
13	19.812	22.362	27.688	28	37.916	41.337	48.278
14	21.064	23.685	29.141	29	39.087	42.557	49.588
15	22.307	24.996	30.578	30	40.256	43.773	50.892

Table 9.3: Critical values of χ^2 for different degrees of freedom and at three levels of significance. The null hypothesis is usually rejected if, for the appropriate number of degrees of freedom, the calculated value of χ^2 is greater than the value tabulated at the $P = 0.05$ significance level.

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Number of pairs of measurements	$P = 0.1$	$P = 0.05$	$P = 0.01$
7	0.714	0.786	0.929
8	0.643	0.738	0.881
9	0.600	0.683	0.833
10	0.564	0.648	0.794
12	0.506	0.591	0.777
14	0.456	0.544	0.715
16	0.425	0.506	0.665
18	0.399	0.475	0.625
20	0.377	0.450	0.591
22	0.359	0.428	0.562
24	0.343	0.409	0.537
26	0.329	0.392	0.515
28	0.317	0.377	0.496
30	0.306	0.364	0.478

Table 9.8: Critical values for the Spearman rank correlation coefficient (r_S) for different numbers of pairs of measurements and at three levels of significance

Note: (i) The null hypothesis is usually rejected if, for the appropriate number of pairs of measurements, the calculated value of r_S is greater than or equal to the value tabulated at the $P = 0.05$ significance level.

(ii) The lower part of Table 9.8 does not have entries for odd numbers of pairs of measurements. Should the data you are analysing comprise (say) 17 pairs of measurements, it is better to err on the side of caution and compare your value of the test statistic with the critical values for 16 pairs rather than those for 18 pairs. Because each critical value for 16 pairs of measurements is higher than the corresponding value for 18 pairs, this makes it less likely that you will mistakenly reject a true null hypothesis.

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Water speed/m s^{-1}	Rank $(R_A)_i$	Number of nymphs	Rank $(R_B)_i$	$D_i = (R_A)_i - (R_B)_i$	D_i^2
0.8	9	35	12	-3	9
1.1	11	28	11	0	0
0.5	5.5	11	6	-0.5	0.25
0.7	7.5	12	7	0.5	0.25
0.2	2.5	7	4	-1.5	2.25
0.4	4	5	1	3	9
0.5	5.5	6	2.5	3	9
1.3	12	21	9	3	9
0.9	10	23	10	0	0
1.7	13	43	13	0	0
0.2	2.5	10	5	-2.5	6.25
0.1	1	6	2.5	-1.5	2.25
0.7	7.5	19	8	-0.5	0.25
				$\sum_{i=1}^{n} D_i = 0$	$\sum_{i=1}^{n} D_i^2 = 47.5$

Table 9.10: Extension of Table 9.9 to include ranks ($(R_A)_i$ and $(R_B)_i$), differences between ranks (D_i) and values of D_i^2

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Degrees of freedom	$P = 0.1$	$P = 0.05$	$P = 0.01$	Degrees of freedom	$P = 0.1$	$P = 0.05$	$P = 0.01$
1	6.314	12.706	63.657	18	1.734	2.101	2.878
2	2.920	4.303	9.925	19	1.729	2.093	2.861
3	2.353	3.182	5.841	20	1.725	2.086	2.845
4	2.132	2.776	4.604	21	1.721	2.080	2.831
5	2.015	2.571	4.032	22	1.717	2.074	2.819
6	1.943	2.447	3.707	23	1.714	2.069	2.807
7	1.895	2.365	3.499	24	1.711	2.064	2.797
8	1.860	2.306	3.355	25	1.708	2.060	2.787
9	1.833	2.262	3.250	26	1.706	2.056	2.779
10	1.812	2.228	3.169	27	1.703	2.052	2.771
11	1.796	2.201	3.106	28	1.701	2.048	2.763
12	1.782	2.179	3.055	29	1.699	2.043	2.756
13	1.771	2.160	3.012	30	1.697	2.042	2.750
14	1.761	2.145	2.977	40	1.684	2.021	2.704
15	1.753	2.131	2.947	60	1.671	2.000	2.660
16	1.746	2.120	2.921	120	1.658	1.980	2.617
17	1.740	2.110	2.898	∞	1.645	1.960	2.576

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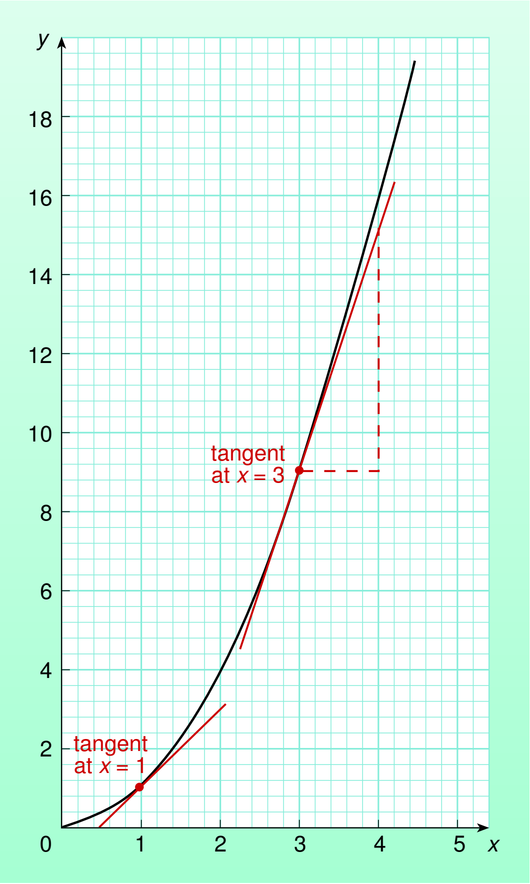
Table 9.13: Critical values of t for the t-test for unmatched samples for different degrees of freedom at three levels of significance. The null hypothesis is usually rejected if the calculated value of t is greater than the value given for the $P = 0.05$ significance level at the appropriate number of degrees of freedom.

Question 10.1 (a)

The gradient of the tangent drawn to the graph of $y = x^2$ at $x = 1$ is

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{(3.0 - 0.0)}{(2.0 - 0.5)} = \frac{3.0}{1.5} = 2.0$$

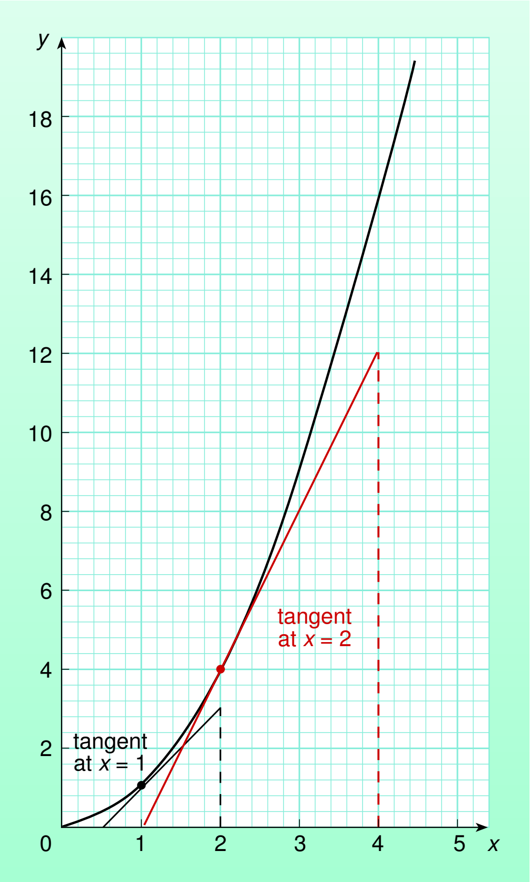
So the gradient of the curve at $x = 1$ is 2.0 to two significant figures.



Question 10.1 (b)

The graph shows $y = x^2$ with a tangent drawn at $x = 2$. The gradient of this tangent is

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{(12.0 - 0.0)}{(4.0 - 1.0)} = \frac{12.0}{3.0} = 4.0$$



So the gradient of the curve at $x = 2$ is 4.0 to two significant figures.

{Note that drawing accurate tangents is difficult; values for the gradient of $y = x^2$ at $x = 2$ found by this method could reasonably be anything between 3.5 and 4.5.

A comparison of the values for gradient at $x = 1$, $x = 2$ and $x = 3$ shows that the gradient increases as x increases. This is consistent with the observed increase in the gradient of the graph as x increases.}

Question 10.2 (a)

$y = x^4$ so $C = 1$ and $n = 4$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \times 4x^3 = 4x^3$$

When $x = 4$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 4 \times 4^3 = 4^4 = 256$

So at $x = 4$ the gradient of the graph is 256.

Question 10.2 (b)

$y = 5x$ so $C = 5$ and $n = 1$ so

$$\frac{dy}{dx} = 5x^{1-1} = 5x^0 = 5$$

The gradient of the graph is 5 for all values of x.

{You may have been able to give this result without differentiating $y = 5x$, from your knowledge of the gradient of straight-line graphs.}

Question 10.2 (c)

$y = 3x^2$ so $C = 3$ and $n = 2$

$$\frac{dy}{dx} = 3 \times 2x^{2-1} = 6x$$

When $x = 4$, $\frac{dy}{dx} = 6 \times 4 = 24$

So at $x = 4$ the gradient of the graph is 24.

Question 10.2 (d)

$y = 5$ so $C = 5$ and $n = 0$

$$\frac{dy}{dx} = 5 \times 0 \times x^{-1} = 0$$

The gradient of the graph is 0 for all values of x.

{You may have been able to give this result without differentiating $y = 5$.}

Question 10.3 (a)

$$y = \frac{1}{\sqrt{x}} = x^{-1/2} \text{ so } C = 1 \text{ and } n = -\tfrac{1}{2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}x^{-1/2-1} = -\frac{x^{-3/2}}{2} = -\frac{1}{2x^{3/2}}$$

This could also be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2x\sqrt{x}}$$

When $x = 4$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2 \times 4 \times \sqrt{4}} = -\frac{1}{2 \times 4 \times 2} = -\frac{1}{16}$$

So at $x = 4$ the gradient of the graph is $-1/16$.

Question 10.3 (b)

$y = \dfrac{2}{x^2} = 2x^{-2}$ so $C = 2$ and $n = -2$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \times (-2)x^{-2-1} = -4x^{-3} = -\frac{4}{x^3}$$

When $x = 4$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{4^3} = -\frac{1}{4^2} = -\frac{1}{16}$$

So at $x = 4$ the gradient of the graph is $-1/16$.

Question 10.4 (a)

$$x = t^7$$

so

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 7t^{7-1} = 7t^6$$

Question 10.4 (b)

$$E = \frac{C}{r} = C\,r^{-1}$$

so

$$\frac{\mathrm{d}E}{\mathrm{d}r} = C \times (-1)\,r^{-1-1} = -Cr^{-2} = -\frac{C}{r^2}$$

Question 10.5

$$z = 4y^2 + y$$

Differentiating each of the terms separately gives

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}y} &= (4 \times 2y^{2-1}) + (1 \times y^{1-1})\\ &= 8y^1 + y^0\\ &= 8y + 1\end{aligned}$$

Question 10.6 (a)

$$x = 2t^3 + 4t^2 - 2t + 3$$

Differentiating this with respect to t gives

$$\frac{dx}{dt} = (2 \times 3t^2) + (4 \times 2t) - 2 = 6t^2 + 8t - 2$$

Differentiating again gives

$$\frac{d^2x}{dt^2} = (6 \times 2t) + 8 = 12t + 8$$

Question 10.6 (b)

$$z = \frac{2}{y} = 2y^{-1}$$

Differentiating with respect to y gives

$$\frac{\mathrm{d}z}{\mathrm{d}y} = 2 \times (-1)y^{-1-1} = -2y^{-2} = -\frac{2}{y^2}$$

Differentiating again gives

$$\frac{\mathrm{d}^2 z}{\mathrm{d}y^2} = -2 \times (-2)y^{-2-1} = 4y^{-3} = \frac{4}{y^3}$$

Question 10.7 (a)

$y = 2\,e^x$ so $C = 2$ and $k = 1$.

$$\frac{dy}{dx} = 2 \times 1\,e^x = 2\,e^x = y \text{ (since } y = 2\,e^x\text{)}.$$

Question 10.7 (b)

$z = e^{t/2}$ so $C = 1$ and $k = \frac{1}{2}$

$$\frac{dz}{dt} = \tfrac{1}{2}\, e^{t/2} = \frac{z}{2} \text{ (since } z = e^{t/2}\text{).}$$

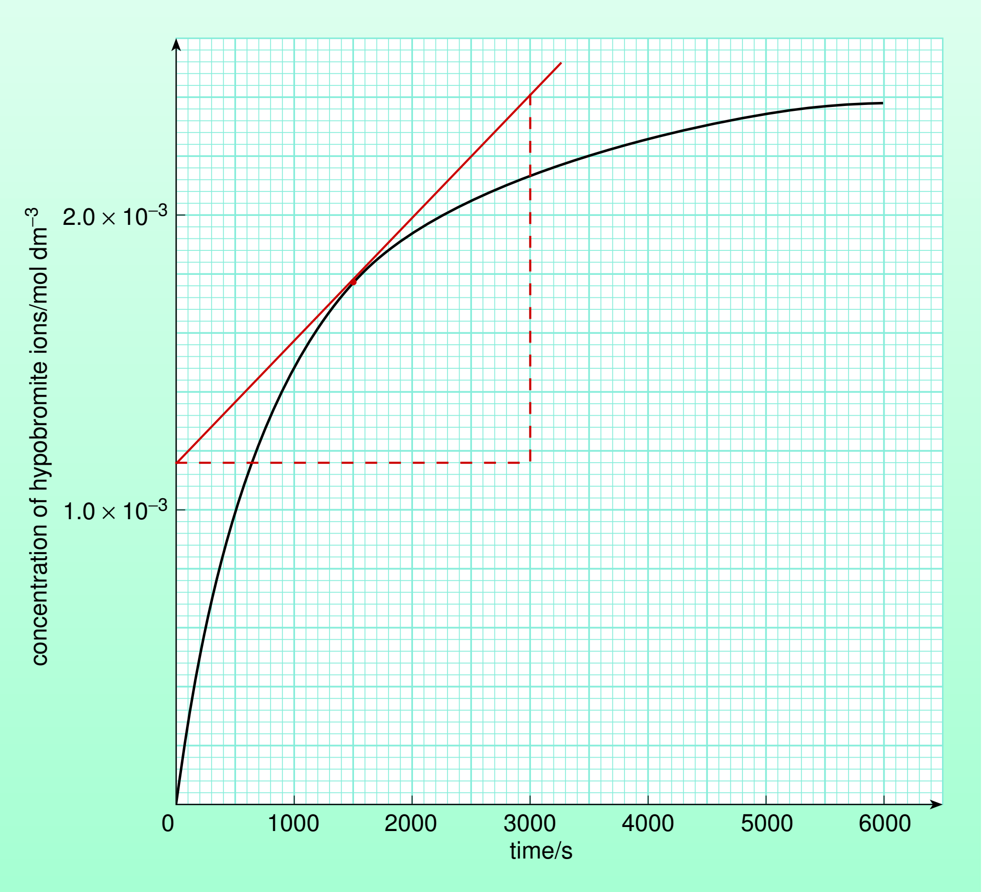


Figure 10.4: A graph to show the increasing concentration of hypobromite ions in a particular chemical reaction, at 25 °C.

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Box 10.4 Differentiating $y = x^2 - 4x + 3$ from first principles

The graph of $y = x^2 - 4x + 3$ is shown again in Figure 10.9 and, as in the previous differentiation from first principles, P and Q represent any two points on the curve.

Since both points lie on the curve, we can say

$$y = x^2 - 4x + 3 \tag{10.1}$$

and

$$(y + \Delta y) = (x + \Delta x)^2 - 4(x + \Delta x) + 3 \tag{10.6}$$

Multiplying out the brackets on the right-hand side of Equation 10.6 gives

$$y + \Delta y = x^2 + 2x\,\Delta x + (\Delta x)^2 - 4x - 4\Delta x + 3$$

and rearranging gives

$$y + \Delta y = (x^2 - 4x + 3) + 2x\,\Delta x - 4\Delta x + (\Delta x)^2$$

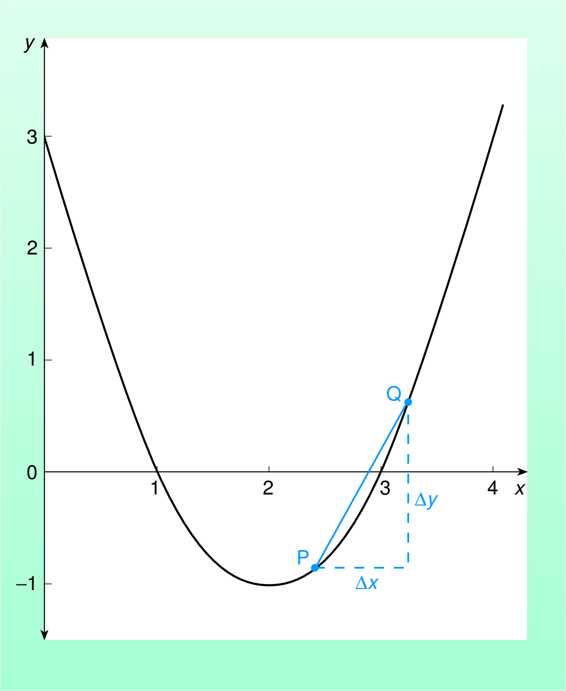

Figure 10.9: Points P and Q on the curve $y = x^2 - 4x + 3$.

Since $y = x^2 - 4x + 3$ (from Equation 10.1), we can subtract y from the left-hand side and $(x^2 - 4x + 3)$ from the right-hand side to give

$$\Delta y = 2x\,\Delta x - 4\Delta x + (\Delta x)^2$$

Dividing both sides by Δx gives

$$\frac{\Delta y}{\Delta x} = 2x - 4 + \Delta x$$

In the limit as Δx approaches zero, the final term on the right-hand side will disappear, and $\frac{\Delta y}{\Delta x}$ will become equal to $\frac{\mathrm{d}y}{\mathrm{d}x}$, so we can say

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 4$$

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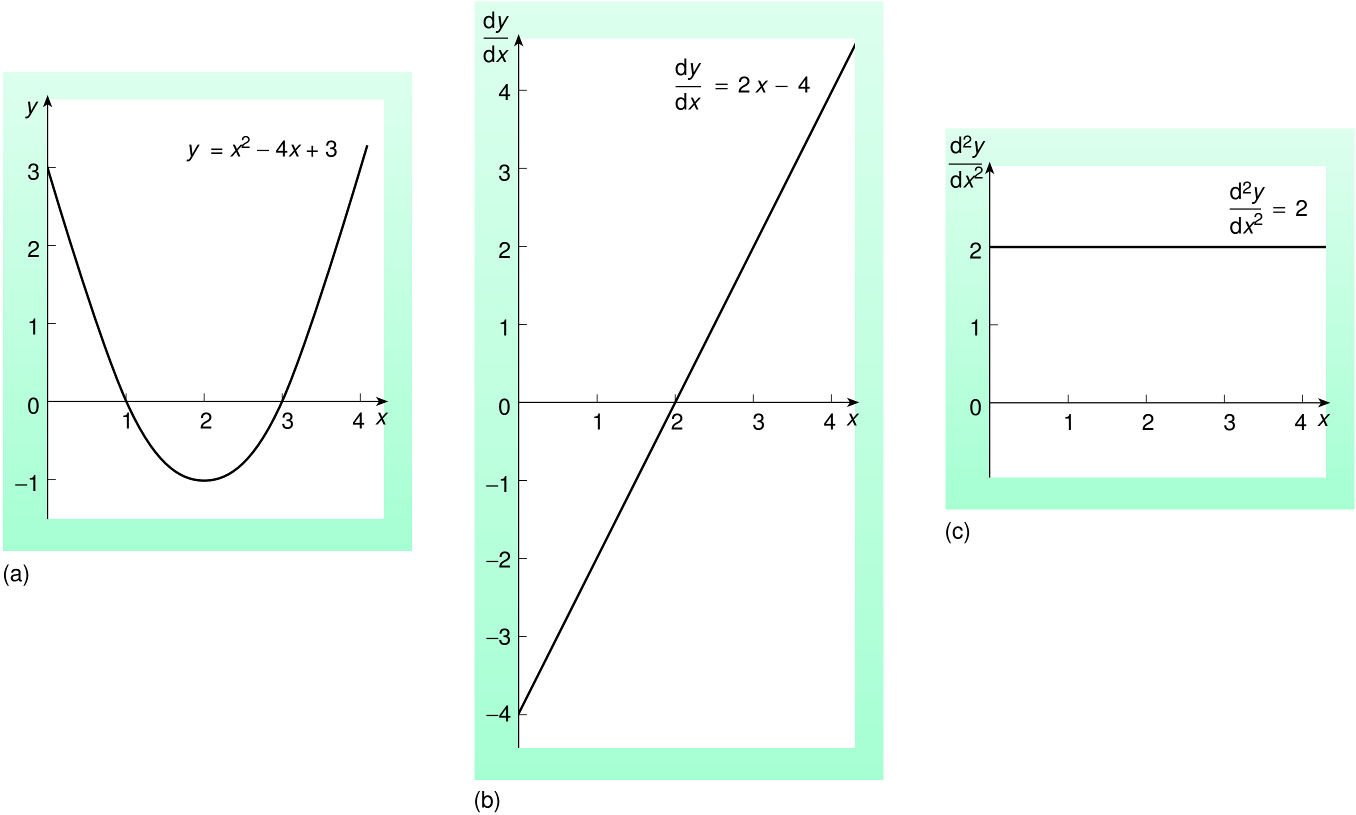

Figure 10.10: Graphs of (a) y against x, (b) $\dfrac{dy}{dx}$ against x and (c) $\dfrac{d^2y}{dx^2}$ against x for $y = x^2 - 4x + 3$.

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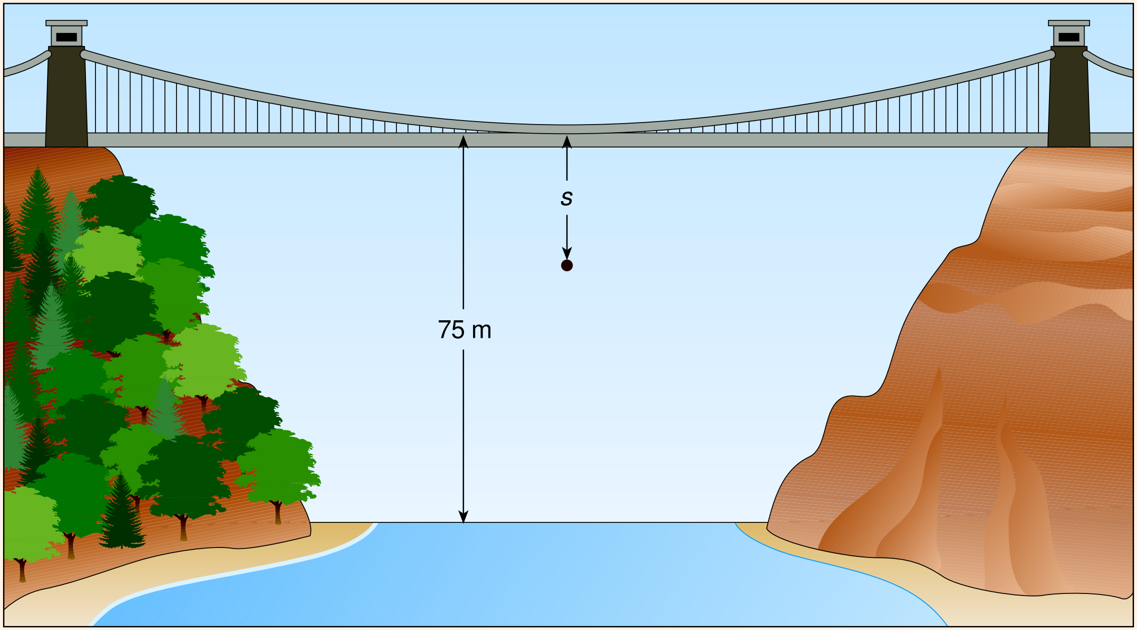

Figure 10.11: An object being dropped from the Clifton Suspension Bridge.

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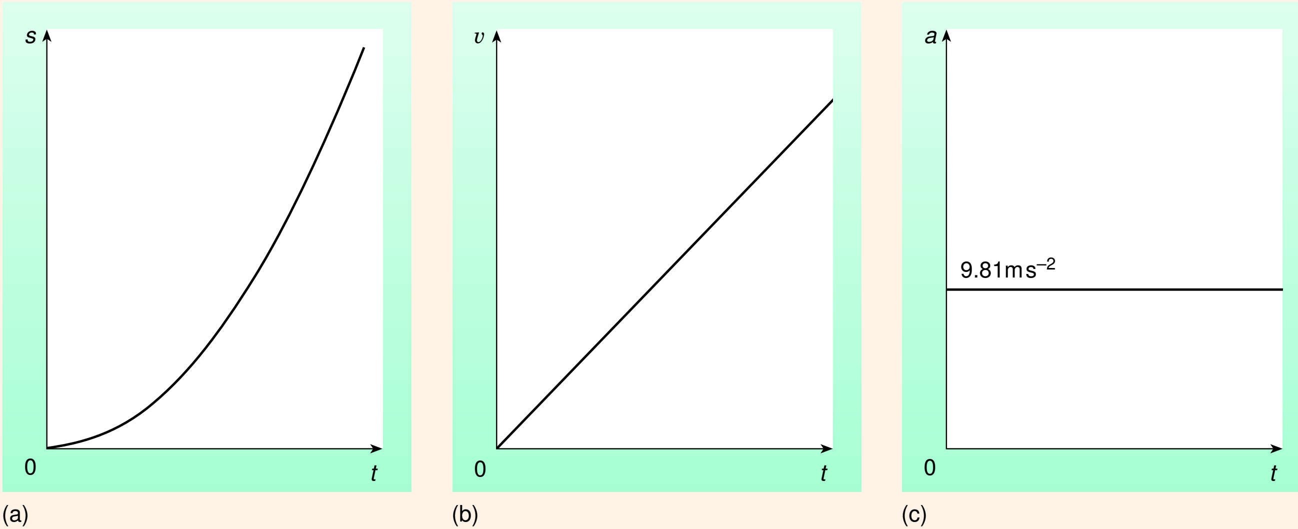

Figure 10.12: Graphs to show the variation of (a) distance, (b) speed and (c) acceleration with time for an object dropped from a bridge. Note that distance from the bridge, speed and acceleration are all measured in a downwards direction.

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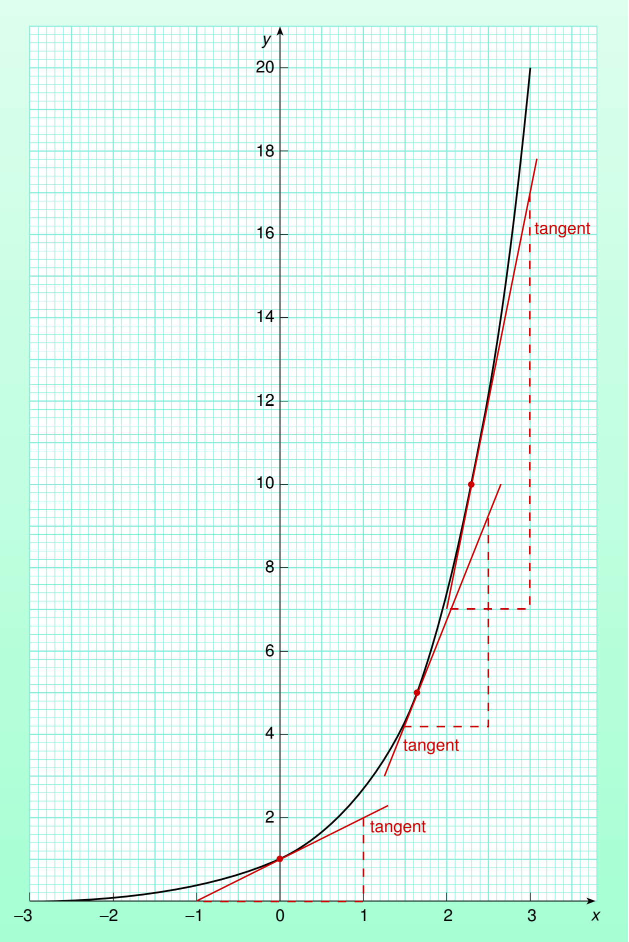

Figure 10.13: A graph of $y = e^x$.

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Question A.1

$$\cos\alpha = \frac{\text{adj}}{\text{hyp}} = \frac{v_x}{v}$$

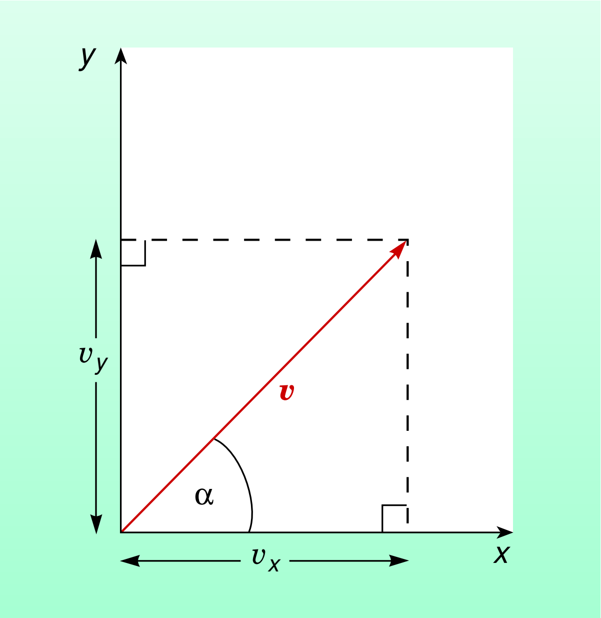
so

$$\begin{aligned} v_x &= v\cos\alpha \\ &= 8.6\ \text{m s}^{-1} \times \cos 42^\circ \\ &= 6.4\ \text{m s}^{-1} \text{ to two significant figures.} \end{aligned}$$

$$\sin\alpha = \frac{\text{opp}}{\text{hyp}} = \frac{v_y}{v}$$

so

$$\begin{aligned} v_y &= v\sin\alpha \\ &= 8.6\ \text{m s}^{-1} \times \sin 42^\circ \\ &= 5.8\ \text{m s}^{-1} \text{ to two significant figures.} \end{aligned}$$



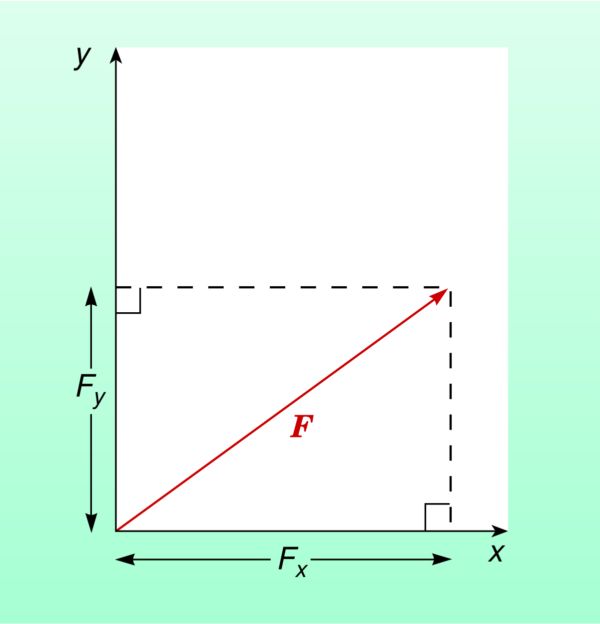
Question A.2

$F^2 = F_x^2 + F_y^2$

So

$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2} \\ &= \sqrt{(4.0\ \mathrm{N})^2 + (3.0\ \mathrm{N})^2} \\ &= 5.0\ \mathrm{N} \end{aligned}$$

$$\begin{aligned} \tan\beta &= \frac{\mathrm{opp}}{\mathrm{adj}} \\ &= \frac{F_y}{F_x} \\ &= \frac{3.0\ \mathrm{N}}{4.0\ \mathrm{N}} \\ &= 0.75 \end{aligned}$$



So $\beta = \tan^{-1}(0.75) = 37°$ to two significant figures.

So the resultant vector $\boldsymbol{F}$ has a magnitude of 5.0 N and acts at an angle of 37° to the horizontal axis.

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